Last week, we computed de Rham cohomology of compact, oriented smooth surfaces. The idea is to use integrals over generators of \( \pi_1(S) \) and surface integrals over all of \( S \). We use Stokes' theorem...

On Friday, we introduced the \( \omega \)-structure. We realized that \( \Omega^1_{\mathbb{C}} \cong \Omega^1_{\mathbb{R}} \oplus \Omega^0 \)

where \( \Omega^1_{\mathbb{R}} \) are \( \omega \)-linear and \( \Omega^0 \) are \( \omega \)-anti-linear.

\[ d \bar{z} = dx + i dy \quad \bar{d} \bar{z} = dx - i dy \]

with \( \Omega^0 \rightarrow \Omega^1 \)

We have \( d = \Delta + \bar{d} \)

and \( \bar{d} = \bar{\Delta} + df \).

If we define

\[ \bar{\Delta} (A \, d \bar{z}) = \frac{\partial A}{\partial z} \, d\bar{z} \wedge d\bar{z} \]

and

\[ \bar{\Delta} (B \, d\bar{z}) = \frac{\partial B}{\partial \bar{z}} \, d\bar{z} \wedge d\bar{z} \]

we obtain maps to 2-forms (anti-comm. diagram \( 2 \bar{\Delta} = -\Delta \)).

Define \( \Omega^{(1,0)} = \) forms \( 1 \)-forms \( \bar{\omega} \)

to be holomorphic if \( \bar{\Delta} \bar{\omega} = 0 \)

(i.e., locally expressible as \( B \, d\bar{z} \) with \( B \) holomorphic).

We might guess that for analysis only \( d \omega \) is interesting, \( \Omega^{0,1} \) doesn't play a role.

But really, we need whole diagram.
What is $\tilde{\omega} d\omega$ in local coordns?

$$\frac{df}{dz} = \frac{1}{2} \left( \frac{df}{dx} - i \frac{df}{dy} \right) \quad \frac{df}{\bar{z}} = \frac{1}{2} \left( \frac{df}{dx} + i \frac{df}{dy} \right)$$

So

$$\frac{\partial}{\partial z} df = \frac{1}{4} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \, dz \wedge d\bar{z}$$

$$= - \frac{\Delta f}{2i} \, dx \wedge dy \quad \Delta : \text{familiar Laplacian whose kernel gives harmonic functions.}$$

(To identify functions with 2-forms $dx \wedge dy$, recover usual defn. of harmonic.

Better: on $\mathbb{R}^2$ this chart valid everywhere, so identification makes sense globally.)

Any theorem about harmonic functions using only local structure of $\mathbb{R}^2$ can be immediately transferred to R.S. setting.

E.g.: 1. If $\phi$ real-valued harmonic function on nbhd $N$ of $x \in X$, then $\exists$ open nbhd $U = N$ of $x$, holom. function $f$ on $U$ s.t. $\phi = \text{Re}(f)$.

2. Maximum principle - if non-const. real-valued harmonic function on connected open set $U \subseteq X$. Then given $x \in U$, $\exists x' \in U$

$$\phi(x') > \phi(x). \quad \text{(consequence of 1 and that holom. maps open)}$$

"Main theorem": $X$ compact R.S., $\rho$ 2-form on $X$

There is a solution $f$ to $\Delta f = \rho \iff \int_X \rho = 0$.

Moreover, the solution $f$ is unique up to addition of a constant.
Why is this the "main theorem"? How does it help to classify Riemann surfaces and their meromorphic functions?

(Remember, using elementary invariant "degree" of holomorphic map, Prop 4.16 of Miranda: \( f_X \) is a compact R.S. with merom. function \( f \) with a single simple pole, then \( X \cong S^2 \).)

For this, need Dolbeaault cohomology.

\[
\begin{align*}
H^{0,0}_X &= \ker \bar{\partial} : \Omega^0 \to \Omega^{0,1} \quad \text{(holomorphic functions)} \\
H^{1,0}_X &= \ker \bar{\partial} : \Omega^{1,0} \to \Omega^2 \quad \text{(holomorphic 1-forms)} \\
H^{0,1}_X &= \text{coker} \bar{\partial} : \Omega^0 \to \Omega^{0,1} \quad \text{(i.e. } \Omega^{0,1} / \text{Im}(\bar{\partial}) \text{)} \\
H^{1,1}_X &= \text{coker} \bar{\partial} : \Omega^{1,0} \to \Omega^2 \quad \text{(i.e. } \Omega^2 / \text{Im}(\bar{\partial}) \text{)}
\end{align*}
\]

2 claims: \( H^{0,1}_X \) is controlling behavior of meromorphic functions on \( X \)

and is computable using the main theorem.

Explore first claim:

When does meromorphic function on \( X \) with simple pole at \( p \)?

there exist a

\[ \text{smooth function with support in } \text{nbd of } p \]

such that \( p \cdot \frac{1}{2} \) is function on \( X \setminus p \).

So finding merom. function with pole at \( p \) \( \iff \) \( \exists \text{ smooth } g \) on \( X \)

s.t. \( g + p \cdot \frac{1}{2} \) holom. on \( X \setminus p \).

Now \( \bar{\partial} \left( p \cdot \frac{1}{2} \right) = (\bar{\partial} p) \cdot \frac{1}{2} \)

1-form with support in \( X \setminus \{ p \} \) since \( p \) loc. const at \( p \).
So we may extend by $0$ at $p$ to give $(0,1)$-forms on $X$, call it $A$.

Want function $g + \beta^{1/2}$, some holom. $g$ on $X \times \mathbb{C}$

i.e. $\bar{\partial} g = - \bar{\partial} (\beta^{1/2}) = - A$, for given $A \in \Omega^{0,1}_X$

Solution $g$ to:

Solution exists $\iff [A] = 0$ in $\text{coker}(\bar{\partial}) = H^{0,1}_X = \Omega^{0,1}_X / \text{Im}(\bar{\partial})$

Simplest case: $H^{0,1}_X = 0$, then true.

Generalize to:

if $\phi$ smooth function on $X \times \mathbb{C}$ which restricts to

d poles as follows:

merom. function with pole at $p$ in some nbhd of $p$,

then for some $\lambda \in \mathbb{C}$, $\phi - \lambda \cdot \beta^{1/2}$ is smooth function on $X$

(extends to)

$\Rightarrow [\bar{\partial} \phi] = \lambda [A]$ in $H^{0,1}_X$.

So given $d$ distinct points $p_1, \ldots, p_d$ with $(0,1)$-forms $A_i$, $i = 1, \ldots, d$

supported on small annuli about $p_i$; then we can find meromorphic

$\phi$ with poles at $p_1, \ldots, p_d$ if $\exists \lambda_i$ s.t. $\lambda_1 [A_1] + \cdots + \lambda_d [A_d]$

$= 0 \in H^{0,1}_X$.

(Automatic if dim $(H^{0,1}_X) < d$)

For second claim: Claim map $\delta : H^{1,0}_X \to \overline{H}^{0,1}_X$ given by $\alpha \mapsto \overline{\alpha}$

(is isomorphism).

Only hard part is surjectivity: Given $[\theta] \in \overline{H}^{0,1}_X$ find $\theta' = \theta + \bar{\partial} f$

s.t. $\bar{\partial} \theta' = 0$ (then $\alpha = \bar{\theta}'$ is holom. $1$-form with $[\theta] = [\theta' + \bar{\partial}(x)]$)

i.e. solve: $\bar{\partial} \delta f = - \bar{\partial} \theta$. Has soln by Main thm +

Stokes' thm.