In the middle of Riemann Mapping Thm pf :

exhibiting $f : \Omega \to B(0,1)$ conformal, bijective

by proving $\exists$ distinguished member of $\mathcal{F}$: all analytic functions g $\Omega \to \mathbb{C}$

with (1) g one-one

(2) $|g(z)| \le 1 \quad \forall z \in \Omega$

(3) $g(z_0) = 0 \quad g'(z_0)$ real, > 0 (some distinguished $z_0 \in \Omega$)

Pick sequence of g's $\in \mathcal{F}$ such that $\lim_n g_n'(z_0) = B$: least upper bound of $g'(z_0)$, $g \in \mathcal{F}$.

If we can prove $\mathcal{F}$ is normal family, then by definition,

$\exists$ subsequence $\{g_{n_k}\} \to f$ uniformly on compacta.

So is $\mathcal{F}$ normal family? At end of last time, showed

$\mathcal{F}$ normal $\iff$ $\mathcal{F}$ totally bounded w.r.t. ρ $\iff$ For every compact $E \subset \Omega$, $\epsilon > 0$

$\exists f_1, \dots, f_n \in \mathcal{F}$ s.t.

For any $f \in \mathcal{F}$

$d(f, f_i) := \sup_{z \in E} d(f(z), f_i(z))$

is $< \epsilon$ for some i.

Working toward stronger equivalence

(Thm. 15 in Ahlfors)

$\iff$ $g \in \mathcal{F}$ are uniformly bounded on every compact set $E \subset \Omega$

$\left(|g(z)| \le M \quad \forall z \in E \right)$
some M

Weaker result acceptable here: Montel's thm (weak version) $g \in \mathcal{F}$ uniformly bounded on Ω 'open' then $\mathcal{F}$ normal

For culture: (strong version) $g \in \mathcal{F}$ omitting two values in $\mathbb{C}$ $\Rightarrow \mathcal{F}$ normal

Key to linking latter two conditions:

Arzela-Ascoli theorem. Recall that a family $\mathcal{F}$ of functions $\Omega \to \mathbb{C}$ is "equicontinuous" on $E \subset \Omega$ if, for every $\epsilon > 0$, $\exists\ \delta > 0$ s.t. $d(f(z), f(z_0)) < \epsilon$ when $|z - z_0| < \delta$ for $z, z_0 \in E$ independent of $f \in \mathcal{F}$.

Theorem: If $f \in \mathcal{F}$ required continuous, then $\mathcal{F}$ is normal iff

(1) $\mathcal{F}$ is equicontinuous for all compact $E \subset \Omega$

(2) for any $z \in \Omega$, the set $\{f(z)\}_{f \in \mathcal{F}}$ is contained in a compact subset of $\mathbb{C}$.

($\Rightarrow$) if $\mathcal{F}$ normal, then $\mathcal{F}$ totally bounded, so given $E \subset \Omega$, pick $f_1, \dots, f_n$ as in previous theorem.

Given any $f \in \mathcal{F}$,
$$d(f(z), f(z_0)) \le d(f(z), f_j(z)) + d(f_j(z), f_j(z_0)) + d(f_j(z_0), f(z_0))$$

these are $< \epsilon$ by choosing f_j corresponding to f.

Since E compact, every continuous function is uniformly continuous (Heine-Borel prop.) so given ϵ pick δ s.t. $|f_j(z) - f_j(z_0)| < \epsilon$ when $|z - z_0| < \delta$ for each $j = 1, \dots, n$

thus $d(f(z), f(z_0)) \le 3\epsilon$.

So $\mathcal{F}$ equicontinuous on any E.

For (2), take $\overline{\{f(z)\}_{f \in \mathcal{F}}}$ (the closure). Show compact by showing Bolzano-Weierstrass prop. (inf. sequence has convergent subseq.)

$\{w_n\}$ sequence in $\overline{\{f(z)\}_{f \in \mathcal{F}}}$. Pick $f_n \in \mathcal{F}$ s.t. $d(f_n(z), w_n) < 1/n$. By normality, $\{f_n\}$ has a convergent subsequence $\{f_{n_k}\}$ and by distance condition $\{w_{n_k}\}$ converges to same value.

($\Leftarrow$) "Cantor's diagonal process"

Let $\{\xi_k\}$ be an everywhere dense sequence of pts. in Ω,

e.g. the collection of cx. numbers in Ω with rational coords (x,y).

Given a sequence $\{f_n\}$ in $\mathcal{F}$, find a subsequence converging at all $\{\xi_k\}$.

Condition (2) guarantees we may do this for any one point ξ_k.

Form an array of indices:

$$n_{1,1} < n_{1,2} < \cdots < n_{1,j} < \cdots$$
$$n_{2,1} < n_{2,2} < \cdots < n_{2,j} < \cdots$$
$$\vdots$$

such that each row is contained in the preceding one (i.e. its elements are a subset of preceding one)

and such that $\lim_{j\to\infty} f_{n_{k,j}}(\xi_k)$ exists for all k.

(just keep taking subsequences of subsequences). Notice $\{n_{j,j}\}$ is strictly increasing by containment condition on rows, and is furthermore a subsequence of each row. Then $\underbrace{\{f_{n_{j,j}}\}}_{\text{"}}$ is the desired subsequence of $\{f_n\}$ converging at all ξ_k.

Rename: $\{f_{n_j}\}$.

Haven't yet used Condition (1): Take E compact $\subset \Omega$. Show $\{f_{n_j}\}$ converge uniformly.

Given $\epsilon > 0$, pick $\delta > 0$ s.t. for $z, z' \in E$, $f \in \mathcal{F}$

then $|z - z'| < \delta \Rightarrow d(f(z), f(z')) < \epsilon/3$ (which can be done by using (1)).

Enough to show Cauchy condition: $\exists\ i_0$ s.t. if $i,j > i_0$ then $d(f_{n_i}(z), f_{n_j}(z)) < \epsilon$ $\forall\ z \in E$.

(since $\mathbb{C}$ complete)

E compact, so cover it with finitely many $\delta/2$ - radius balls.

Pick ξ_k for each such ball. Choose i_0 s.t. if $i,j > i_0$, then

$d(f_{n_i}(\xi_k), f_{n_j}(\xi_k)) < \epsilon/3$ for all ξ_k : representatives from $\delta/2$-balls.

Then $d(f_{n_i}(z), f_{n_j}(z)) < \underbrace{d(f_{n_i}(z), f_{n_i}(\xi_k))}_{\xi_k \text{ chosen in } \delta/2\text{-ball of } z} + d(f_{n_i}(\xi_k), f_{n_j}(\xi_k)) + d(f_{n_j}(\xi_k), f_{n_j}(z))$

$< \epsilon.$ //

Finally, we arrive at a characterization of normality we can use:

Theorem: A family $\mathcal{F}$ consisting of <u>analytic</u> functions is normal if and only if $\mathcal{F}$ is uniformly bounded on every compact set.

($\Leftarrow$) uniformly bdd. on compact set immediately implies (2) in Arzela-Ascoli thm.

Must show equicontinuity. Let C: boundary of closed disk of radius r in Ω.

By Cauchy Integral formula: For z, z_0 inside disk:

$$f(z) - f(z_0) = \frac{1}{2\pi i} \int_C \left(\frac{1}{\xi - z} - \frac{1}{\xi - z_0} \right) f(\xi)\, d\xi$$

$$= \frac{z - z_0}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)(\xi - z_0)}.$$

If $|f| \le M$ on C and we restrict z, z_0 away from boundary C of disk, say that they lie in disk with radius $r/2$.

then $\Rightarrow$ $$|f(z) - f(z_0)| \le \frac{4M|z - z_0|}{r}$$

By our assumption, can choose M valid for all $f \in \mathcal{F}$

so get equicontinuity on any disk of radius $r/2$ when disk of radius $r \subset \Omega$.

Now use disks to cover a compact $E \subset \Omega$. Get finite subcover, ...