In the middle of Riemann Mapping Thm. 15:

exhibiting \( f : \Omega \to B(0, 1) \) conformal, bijective

by proving \( \exists \) distinguished member of \( \mathcal{F} \) of all functions \( g : \Omega \to \mathbb{C} \) with

1. \( g \) one-one
2. \( |g(z)| \leq 1 \) \( \forall z \in \Omega \)
3. \( g(z_0) = 0 \), \( g'(z_0) \) real, \( > 0 \) \( (\text{some distinguished } z_0 \in \Omega) \)

Pick sequence of \( g' \)'s \( \in \mathcal{F} \) such that

\[
\lim_{n \to \infty} g'_n(z_0) = \beta : \text{least upper bound of } g'(z_0), \forall \mathcal{F}
\]

If we can prove \( \mathcal{F} \) is normal family, then by definition,

\( \exists \) subsequence \( \{g_{n_k}\} \to f \) uniformly on compacta.

So is \( \mathcal{F} \) normal family? At end of last time, showed

\( \mathcal{F} \) normal \( \iff \mathcal{F} \) totally bounded \( \iff \) For every compact \( E \subset \Omega \)

\( g \in \mathcal{F} \) are uniformly bounded on every compact set \( E \subset \Omega \)

\[
\left( |g(z)| \leq M \ \forall z \in E \right) \text{ (some } M \text{)}
\]

Weaker result acceptable here: Montel's thm (weak version) \( g \in \mathcal{F} \) uniformly bounded on \( \Omega \) then \( \mathcal{F} \) normal

For culture: (strong version) \( g \in \mathcal{F} \) omitting two values in \( \mathbb{C} \) \( \Rightarrow \) \( \mathcal{F} \) normal
Key to linking latter two conditions:

**Arzelà-Ascoli theorem.** Recall that a family $F$ of functions $Ω → C$

is "equicontinuous" on $E ⊆ Ω$ if, for every $ε > 0$, there exists $δ > 0$ such that $d(f(z), f(z₀)) < ε$ when $|z - z₀| < δ$ for $z, z₀ ∈ E$ independent of $f ∈ F$.

**Theorem:** If $f ∈ F$ required continuous, then $F$ is normal iff

1. $F$ is equicontinuous for all compact $E ⊆ Ω$.
2. For any $z₀ ∈ Ω$, the set $\{ f(z₀) | f ∈ F $ is contained in a compact subset of $C$.

($⇒)$ if $F$ normal, then $F$ totally bounded, so given $E ⊆ Ω$, pick $f₁, ..., fₙ$ as in previous theorem.

Given $f ∈ F$, $d(f(z), f(z₀)) ≤ d(f(z), f_j(z)) + d(f_j(z), f(z₀))$

$≤ ε$ by choosing $f_j$ corresponding to $f$.

Since $E$ compact, every continuous function is uniformly continuous (Heine-Borel prop.)

thus $d(f(z), f(z₀)) ≤ 3ε$.

so $F$ equicontinuous on any $E$.

For (2), take $\{ f(z) \}_f ∈ F$ (the closure). Show compact by showing Bolzano-Weierstrass prop. (inf sequence has convergent subseq.)

$\{ wₙ \}$ sequence in $\{ f(z) \}_f ∈ F$. Pick $fₙ ∈ F$ s.t.

$d(fₙ(z), wₙ) < 1/n$. By normality, $\{ fₙ \}$ has a convergent subsequence $\{ fₙₖ \}$

and by distance condition $\{ wₙₖ \}$ converges to some value.
"Cantor's diagonal process."

Let \( \xi_{\mathbb{S}k} \) be an everywhere dense sequence of pts. in \( \Omega \), e.g., the collection of \( \mathbb{R} \) numbers in \( \Omega \) with rational coords \((x,y)\).

Given a sequence \( \xi_{\mathbb{F}n} \) in \( \mathbb{F} \), find a subsequence converging at all \( \xi_{\mathbb{S}k} \).

Condition (2) guarantees we may do this for any one point \( \xi_k \).

Form an array of indices:

\[
\begin{align*}
n_{1,1} & < n_{1,2} < \cdots < n_{1,j} < \cdots \\
n_{2,1} & < n_{2,2} < \cdots < n_{2,j} < \cdots \\
& \vdots
\end{align*}
\]

Such that each row is contained in the preceding one (i.e., its elements are a subset of the preceding one) and such that \( \lim_{j \to \infty} f_{\mathbb{F}n_{kj}}(\xi_k) \) exists for all \( k \).

(Just keep taking subsequences of subsequences.) Notice \( \xi_{\mathbb{F}n_{kj}} \) is strictly increasing by containment condition on rows, and is furthermore a subsequence of each row. Then \( \xi_{\mathbb{F}n_{kj}} \) is the desired subsequence of \( \xi_{\mathbb{F}n} \) converging at all \( \xi_k \). Rename: \( \xi_{\mathbb{F}n_j} \).

Haven't yet used Condition (1): Take \( E \) compact \( \subset \Omega \). Show \( \xi_{\mathbb{F}n_j} \) converge uniformly.

[Given \( \varepsilon > 0 \), pick \( \delta > 0 \) s.t. for \( z, z' \in E \), \( f \in \mathbb{F} \)

\[\text{then } |z - z'| < \delta \Rightarrow d\left(f(z), f(z')\right) < \varepsilon / 3 \] (which can be done by using (1)).

Enough to show Cauchy condition: \( \exists i_0 \text{ s.t. if } |i| > i_0 \text{ then } d\left(f_{\mathbb{F}n_i}(z), f_{\mathbb{F}n_j}(z)\right) < \varepsilon \) if \( z \in E \).

(since \( E \) complete)
$E$ compact, so cover it with finitely many $\frac{1}{2}$-radius balls.

Pick $S_k$ for each such ball. Choose $i_0$ s.t. if $i,j \geq i_0$, then
\[ d(f_{n_i}(S_k), f_{n_j}(S_k)) < \frac{\varepsilon}{3} \text{ for all } S_k \text{ representatives from } \frac{1}{2}\text{-balls}. \]

Then
\[ d(f_{n_i}(z), f_{n_j}(z)) < d(f_{n_i}(z), f_{n_i}(S_k)) + d(f_{n_i}(S_k), f_{n_j}(S_k)) + d(f_{n_j}(S_k), f_{n_j}(z)) \]
\[ S_k \text{ chosen in } \frac{1}{2}\text{-ball of } z \]
\[ < \varepsilon. \]

\[ \text{\underline{\text{Finally, we arrive at a characterization of normality we can use:}}}
\]

\textbf{Theorem:} A family $F$ consisting of analytic functions is normal if and only if $F$ is uniformly bounded on every compact set.

$(\Rightarrow)$ uniformly held on compact set immediately implies (2) in Arzela-Ascoli theorem.

Must show equicontinuity. Let $C$: boundary of closed disk of radius $r$ in $\Omega$.

By Cauchy integral formula: For $z, z_0$ inside disk:
\[ f(z) - f(z_0) = \frac{1}{2\pi i} \int_C \left( \frac{1}{z - \xi} - \frac{1}{z_0 - \xi} \right) f(\xi) \, d\xi \]
\[ = \frac{z - z_0}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)(\xi - z_0)}. \text{ If } |f| \leq M \text{ on } C \]
\[ \text{and we restrict } z, z_0 \text{ away from boundary } C \text{ of disk say that they lie in disk with radius } 1/2. \]

Then
\[ |f(z) - f(z_0)| \leq \frac{4M|z - z_0|}{r} \]

By our assumption, can choose $M$ valid for all $f \in F$.

So get equicontinuity on any disk of radius $r/2$ when disk of radius $r \in \Omega$. Now use disks to cover a compact $E \subset \Omega$. Get finite subcover, ...