Find useful condition to prove $F$ normal. $F = \{ f : \Omega \to \mathbb{C} \}$

Last class $F$ normal $\iff$ $F$ totally bounded in $\rho$-metric

$\iff$ (1) $F$ equicontinuous $\forall E \subset \Omega$ compact

Supposing $f \in F$ continuous (Arzelà-Ascoli)

(2) For any $z \in \Omega$, $\{ f(z) \}_{f \in F}$ is contained in compact subset of $\mathbb{C}$.

$\iff$ $F$ is uniformly bounded on every compact set $E \subset \Omega$.

$\Rightarrow$ $|f(z)| \leq M_z \quad \forall z \in E \quad \forall f \in F$

($\Rightarrow$ ) (2) implies $|f(z)| \leq M_z \quad \forall z \in E$ for fixed $f \in F$. We write $M_z$ to emphasize that bound may depend on $z$.

But then (1) - equicontinuity - implies $F$ is c.i. (2 - fixed)

$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon \quad \forall f \in F$.

$\Rightarrow$ $|f(z)| < M_{z_0} + \epsilon \quad \forall f \in F$.

Any compact $E$ can be covered by finitely many such neighborhoods so $f \in F$ uniformly bounded.
E compact, so cover it with finitely many \( \frac{1}{2} \)-radius balls.

Pick \( \delta_k \) for each such ball. Choose \( i_0 \) s.t. if \( i,j > i_0 \), then

\[
d( f_{i_0} (\delta_k), f_{i_0} (\delta_k) ) < \frac{\epsilon}{3} \quad \text{for all } \delta_k \text{ representatives from } \frac{1}{2} \text{-balls.}
\]

Then

\[
d( f_{i_0} (z), f_{i_0} (z) ) < d( f_{i_0} (z), f_{i_0} (\delta_k) ) + d( f_{i_0} (\delta_k), f_{i_0} (\delta_k) ) < \frac{\epsilon}{3}
\]

if \( \delta_k \) chosen in \( \frac{1}{2} \)-ball of \( z \)

\[
< \epsilon.
\]

\[\] Finally, we arrive at a characterization of normality we can use:

**Theorem:** A family \( \mathcal{F} \) consisting of analytic functions is normal if and only if

\( \mathcal{F} \) is uniformly bounded on every compact set.

(\( \Leftarrow \)) uniformly bdd. on compact set immediately implies (2) in Arzela-Ascoli theorem.

Must show equicontinuity. Let \( C \): boundary of closed disk of radius \( r \) in \( \Omega \).

By Cauchy integral formula: For \( z \neq z_0 \) inside disk:

\[
\frac{1}{2\pi i} \int_C \left( \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - z_0} \right) d\zeta
\]

then

\[
\Rightarrow |f(z) - f(z_0)| \leq \frac{4M |z - z_0|}{2\pi r} \quad \text{if } |f| \leq M \text{ on } C
\]

and we restrict \( z, z_0 \) away from boundary \( C \) of disk say that they lie in disk with radius \( \frac{r}{2} \).

By our assumption, can choose \( M \) valid for all \( f \in \mathcal{F} \)

so get equicontinuity on any disk of radius \( \frac{r}{2} \) when disk of radius \( r \) in \( \Omega \).

Now use disks to cover a compact \( E \subset \Omega \). Get finite subcover, ...
The theory of normal families \( \Rightarrow \) \( \exists \) subsequence of \( g_n \), \( g_n \in F \) converging uniformly on compacts to \( f \).

Need to show \( f \in F \). It is clear

that several properties are preserved in limit: \( f \) analytic,

\[
|f(z)| \leq 1 \text{ in } \Omega, \quad f(z_0) = 1 \quad (\text{closed conditions}). \quad \text{Moreover,}
\]

\[
f'(z_0) = B \quad \text{by construction, where } B \text{ denoted } \lim_{n \to \infty} g_n'(z_0).
\]

Remains to check: \( f \) is one-one to show \( f \in F \).

Pick \( z_1 \in \Omega \). Then \( \hat{g}(z) = g(z) - g(z_1) \) for each \( g \in F \) yields

family that is \( \neq 0 \) for all \( z \in \Omega \setminus \{z_1\} \) since \( g \)'s were one-one.

Now \( \hat{g}_n \to f(z) - f(z_1) \).

Hence, \( \hat{g}_n \) converges to \( f(z) - f(z_1) \) and \( \hat{g}_n \neq 0 \) in region \( \Omega \setminus \{z_1\} \).

Then either \( f(z) - f(z_1) \equiv 0 \ \forall \ z \in \Omega \setminus \{z_1\} \) or \( f(z) - f(z_1) \neq 0 \ \forall \ z \in \Omega \setminus \{z_1\} \).

Now \( f(z) \equiv f(z_1) \), i.e. \( f \) constant, is not possible since, for example,

\[
f'(z_0) = B > 0 \quad \text{a positive real number}.
\]

So must be that \( f(z) \neq f(z_1) \ \forall \ z \in \Omega \setminus \{z_1\} \). Since \( z_1 \) was

arbitrary, this proves \( f \) is one-one.

\(*\) (Hurwitz thm is combination of Cauchy integral formula + isolated zeros. See p.178 of Ahlfors.)
Lastly, we must show this \( f \) with maximal derivative at \( z_0 \) is onto the open ball \( B_{1} \) \( \forall w: |w| < 1 \).

Suppose that \( f(z_0) \notin B_{1} \) with \( f(z_0) = w_0 \). Construct \( G \in F \) with \( G'(z_0) > B \) (contradicting maximality of \( f'(z_0) \)).

Now \( w_0 \neq 0 \) by assumption that \( f(z_0) = 0 \). Map \( B_{1} \) to itself, taking distinguished point in \( z_2 \):
\[
W \rightarrow \frac{W - W_0}{1 - \overline{W_0}W}
\]

This is accomplished by the linear fractional transformation:
\[
\Psi(z) := \frac{f(z) - W_0}{1 - \overline{W_0}f(z)}.
\]

Then since \( f(z) \) omits \( W_0 \), \( \Psi \) is one-one, analytic function:
\[
\Omega \rightarrow \text{Ann}(0,1) = \{ z \mid 0 < |z| < 1 \}
\]

Can define \( \sqrt{\Psi(z)} \) since we can define \( \log(\Psi(z)) \) by path integration of \( \frac{\Psi'(z)}{\Psi(z)} \).

This is well-defined independent of path by Cauchy integral theorem, since \( \Omega \) simply-connected.

As we saw before, there are distinct branches \( h(z) \), \(-h(z)\), so:
\[
h(z)^2 = \Psi(z)
\]
with \( 0 < |h(z)| < 1 \).

So \( \Psi(z) \) one-one \( \Rightarrow \) \( h(z) \) one-one.

We do one last linear transformation to normalize \( h(z) \), and make it \( -0 \) at \( z_0 \):
\[
G(z) := \frac{h(z) - h(z_0)}{1 - \overline{h(z_0)}h(z)} \left( \frac{1}{h'(z_0)} \right) \text{. Now } G \in F.
\]
Then we just calculate $G'(z_0)$ using many applications of the chain rule:

\[
G'(z_0) = \frac{|h'(z_0)|}{1 - |h(z_0)|^2} = \frac{1 + |w_0|}{2\sqrt{w_0}} B > B,
\]

our desired contradiction.

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Short facts about boundaries:

Given two regions $\Omega_1$, $\Omega_2$, boundaries $\partial(\Omega_1)$

Suppose $f$ maps $\Omega_1$ conformally to $f(\Omega_1)$.

If $f(\Omega_1)$ has boundary $\partial(\Omega_2)$ and $\exists z_0 \in \Omega_1$

st. $f(z_0) \in \Omega_2$, then $f(\Omega_1) = \Omega_2$.

(so image of conformal map deleted by boundary + one pt.)

If: By definition, regions are open, connected $f(\Omega_1)$ open, say by inverse function

\[
\Rightarrow f(\Omega_1) \text{ either in } \Omega_2 \\
\text{or } \Omega_2 \setminus (\Omega_2 \cup \partial(\Omega_2))
\]

But since $f(z_0) \in \Omega_2$, must be open set in $\Omega_2$.

Show $f(\Omega_1)$ closed relative to $\Omega_2$, hence $= \Omega_2$:

clear since $\partial(f(\Omega_1)) = \partial(\Omega_2)$ disjoint from $\Omega_2$

so $f(\Omega_1) \cap \Omega_2 = f(\Omega_1)$. √
The (Osgood-Carathéodory) theorem states that if $\Omega_1, \Omega_2$ are bounded, simply connected regions with $\partial(\Omega_i)$ simple, closed curves, then a conformal map $f: \Omega_1 \to \Omega_2$ can be extended to a continuous, bijective map:

$$\Omega_1 \cup \partial \Omega_1 \to \Omega_2 \cup \partial \Omega_2.$$