Find useful condition to prove $\mathcal{F}$ normal. $\mathcal{F} = \{ f : \Omega \to \mathbb{C} \}$

Last class $\quad \mathcal{F}$ normal $\iff$ $\mathcal{F}$ totally bounded in ρ-metric

$\iff$ (1) $\mathcal{F}$ equicontinuous $\forall E \subset \Omega$ compact

Supposing $f \in \mathcal{F}$ continuous (Arzela–Ascoli)

(2) For any $z \in \Omega$, $\{ f(z) \}_{f \in \mathcal{F}}$ is contained in compact subset of $\mathbb{C}$.

$\iff$ $\mathcal{F}$ is uniformly bounded on every compact set. $E \subset \Omega$

suppose $f \in \mathcal{F}$ analytic

$\left(|f(z)| \le M \quad \forall z \in E, \ \forall f \in \mathcal{F} \right)$

$(\Rightarrow)$ (2) implies for fixed $z \in \Omega$ $\quad |f(z)| \le M_z \quad \forall f \in \mathcal{F}$. We write M_z to emphasize that bound may depend on z.

But then (1) – equicontinuity – implies $\exists \delta$ s.t. (z_0 fixed)

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon \quad \forall f \in \mathcal{F}.$$

$$\Rightarrow |f(z)| < M_{z_0} + \epsilon \quad \forall f \in \mathcal{F}.$$

Any compact E can be covered by finitely many such neighborhoods

so $f \in \mathcal{F}$ uniformly bounded.

E compact, so cover it with finitely many $\delta/2$ - radius balls.

Pick ξ_k for each such ball. Choose i_0 s.t. if $i,j > i_0$, then

$d(f_{n_i}(\xi_k), f_{n_j}(\xi_k)) < \epsilon/3$ for all ξ_k : representatives from $\delta/2$ - balls.

Then $d(f_{n_i}(z), f_{n_j}(z)) < d(f_{n_i}(z), f_{n_i}(\xi_k)) + d(f_{n_i}(\xi_k), f_{n_j}(\xi_k)) + d(f_{n_j}(\xi_k), f_{n_j}(z))$

ξ_k chosen in $\delta/2$ - ball of z

$< \epsilon$. //

Finally, we arrive at a characterization of normality we can use:

Theorem: A family $\mathcal{F}$ consisting of analytic functions is normal if and only if $\mathcal{F}$ is uniformly bounded on every compact set.

($\Leftarrow$) uniformly bdd. on compact set immediately implies (2) in Arzela-Ascoli thm.

Must show equicontinuity. Let C: boundary of closed disk of radius r in Ω.

By Cauchy Integral formula: For z, z_0 inside disk:

$$f(z) - f(z_0) = \frac{1}{2\pi i}\int_C \left(\frac{1}{\xi - z} - \frac{1}{\xi - z_0}\right) f(\xi)\, d\xi$$

$$= \frac{z - z_0}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)(\xi - z_0)}.$$

If $|f| \le M$ on C and we restrict z, z_0 away from boundary C of disk say that they lie in disk with radius $r/2$.

then $\Rightarrow$ $$|f(z) - f(z_0)| \le \frac{4M|z - z_0|}{r}$$

By our assumption, can choose M valid for all $f \in \mathcal{F}$

so get equicontinuity on any disk of radius $r/2$ when disk of radius $r \subseteq \Omega$.

Now use disks to cover a compact $E \subset \Omega$. Get finite subcover, ...

The theory of normal families $\Rightarrow$ $\exists$ subsequence of $\{g_n\}$, $g_n \in \mathcal{F}$

converging uniformly on compacta to f.

Need to show $f \in \mathcal{F}$. It is clear

that several properties are preserved in limit: f analytic,

$|f(z)| \leq 1$ in Ω, $f(z_0) = 1$ (closed conditions). Moreover,

$f'(z_0) = B$ by construction, where "B" denoted $\lim\limits_{n\to\infty} g_n'(z_0)$.

$\left(\sup\limits_{g \in \mathcal{F}} g'(z_0)\right)$

<u>Remains</u> to check: f is one-one to show $f \in \mathcal{F}$.

Pick $z_1 \in \Omega$. Then $\tilde{g}(z) := g(z) - g(z_1)$ for each $g \in \mathcal{F}$ yields

family that is $\neq 0$ for all $z \in \Omega \setminus \{z_1\}$ since g's were one-one.

Now $\tilde{g}_{n_k} \to f(z) - f(z_1)$.

<u>Hurwitz' thm</u>* ~~thm~~ if $\tilde{g}_{n_k} \to f(z) - f(z_1)$ and $\tilde{g}_{n_k} \neq 0$ in region $\Omega \setminus \{z_1\}$

then either $f(z) - f(z_1) \equiv 0$ $\forall z \in \Omega \setminus \{z_1\}$ or $f(z) - f(z_1) \neq 0$ $\forall z \in \Omega \setminus \{z_1\}$.

Now $f(z) \equiv f(z_1)$, i.e. f constant, is not possible since, for example,

$f'(z_0) = B > 0$, a <u>positive</u> real number.

So must be that $f(z) \neq f(z_1)$ $\forall z \in \Omega \setminus \{z_1\}$. Since z_1 was

arbitrary, this proves f is one-one.

* (Hurwitz thm is combination of Cauchy integral formula + isolated zeroes. See p.178 of Ahlfors.)

Lastly, we must show this f with maximal derivative at z_0 is onto the open ball $\{w : |w| < 1\}$.

Suppose that $\exists\ w_0 \in B(0,1)$ with $f(\Omega) \not\ni w_0$. Construct $G \in \mathcal{F}$ with $G'(z_0) > B$ (contradicting maximality of $f'(z_0) = B$)

Now $w_0 \neq 0$ by assumption that $f(z_0) = 0$.
distinguished point in Ω

Map $B(0,1)$ to itself, taking $w_0 \to 0$:

This is accomplished by the linear fractional transformation: $w \mapsto \dfrac{w - w_0}{1 - \overline{w_0} w}$

Set $$\psi(z) := \frac{f(z) - w_0}{1 - \overline{w_0} f(z)}.$$

Then since $f(\Omega)$ omits w_0, ψ : one-one, analytic function $\Omega \to \text{Ann}(0,1) = \{ z \mid 0 < |z| < 1 \}$

Can define $\sqrt{\psi(z)}$ since we can define $\log(\psi(z))$ by path integration of $\psi'(z)/\psi(z)$
this is well-defined indep. of path by Cauchy integral thm, since Ω simply-conn.

As we saw before, there are distinct branches $h(z)$, $-h(z)$ s.t. $h(z)^2 = \psi(z)$ with $0 < |h(z)| < 1$

So $\psi(z)$ one-one $\Rightarrow$ $h(z)$ one-one.

We do one last linear transformation to normalize $h(z)$, and make it $= 0$ at z_0:

$$G(z) := \frac{h(z) - h(z_0)}{1 - \overline{h(z_0)}\, h(z)} \cdot \left(\frac{|h'(z_0)|}{h'(z_0)} \right).$$ Now $G \in \mathcal{F}$.

Then we just calculate $G'(z)$ using many applications of the chain rule, get

$$G'(z_0) = \frac{|h'(z_0)|}{1 - |h(z_0)|^2} = \frac{1 + |w_0|}{2\sqrt{w_0}} B > B,$$

our desired contradiction.

Short facts about boundaries:

Given two regions Ω_1, Ω_2, boundaries $\partial(\Omega_i)$

Suppose f maps Ω_1 conformally to $f(\Omega_1)$.

If $f(\Omega_1)$ has boundary $\partial(\Omega_2)$ and $\exists\, z_0 \in \Omega_1$

s.t. $f(z_0) \in \Omega_2$, then $f(\Omega_1) = \Omega_2$.

(so image of conformal map det'd by boundary + one pt.)

Pf: By definition, regions are open, conn. $f(\Omega_1)$ open, say by inverse function thm since $f' \neq 0$ on Ω_1, and connected (since f continuous)

$\Rightarrow$ $f(\Omega_1)$ either in Ω_2 or $\mathbb{C} \setminus (\Omega_2 \cup \partial(\Omega_2))$

But since $f(z_0) \in \Omega_2$, must be open set in Ω_2.

Show $f(\Omega_1)$ closed relative to Ω_2, hence $= \Omega_2$:

clear since $\partial(f(\Omega_1)) = \partial(\Omega_2)$ disjoint from Ω_2

so $\overline{f(\Omega_1)} \cap \Omega_2 = f(\Omega_1)$. ✓

Thm (Osgood-Caratheodory) Ω_1, Ω_2 bounded, simply conn. regions with $\partial(\Omega_i)$ simple, closed curves (continuous), then conformal map $f: \Omega_1 \to \Omega_2$ can be extended to a continuous; bijective map:

$$\Omega_1 \cup \partial\Omega_1 \longrightarrow \Omega_2 \cup \partial\Omega_2 .$$