linear fractional transformations (Anfs 33)

\[
SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, \ ad - bc = 1 \right\}
\]

Matrix group acting on points in \( \mathbb{C} \) by \( T \), with \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \),

\[
T : z \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{az + b}{cz + d}.
\]

(just matrix mul: \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \))

Which makes clear this is a group action where composition corresponds to matrix multiplication.

Also consider \( GL(2, \mathbb{C}) \) with \( ad - bc \neq 0 \).

(Note if \( ad - bc = 0 \), then rows are multiples of each other \( \Rightarrow \) transform is constant.

These obviously behave differently. Want to rule them out.)

Note \( GL(2, \mathbb{C}) \) and \( SL(2, \mathbb{C}) \) differ by matrices of form \( \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \),

which is just a scaling \( z \mapsto kz \).

Think \( k \) and \( \text{arg} \) (or \( \text{angle} \)).

Finally, \( T(a) := \frac{a}{c} \) useful convention (or extension to Riemann sphere)

\[
T(-\frac{d}{c}) := \infty.
\]
Thus \( T \) maps Riemann sphere \( \mathbb{C} \cup \{\infty\} \) continuously to itself with continuous inverse. Or \( \mathbb{C} \setminus \left\{ \frac{-d}{c} \right\} \leftrightarrow \mathbb{C} \setminus \left\{ \frac{a}{c} \right\} \) bij.

\( T, T' \) are conformal since \( z = T(T^{-1}(z)) = T^{-1}(T(z)) \)

and so \( T'(z) \) non-zero by chain rule.

Every matrix in \( \text{GL}(2, \mathbb{C}) \) can be decomposed as:

\[
\begin{pmatrix}
1 & \alpha \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
k_1 \\
k_2
\end{pmatrix}
\] or

\[
\begin{pmatrix}
1 & \alpha_1 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
k_1 \\
k_2
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

(Bruhat decomposition)

so only need to understand elementary transformations:

\[
\begin{pmatrix}
1 & d \\
0 & 1
\end{pmatrix} : z \mapsto z + d
\]

(translation by vector \( d \))

\[
\begin{pmatrix}
k & 0 \\
0 & 1
\end{pmatrix} : z \mapsto k z
\]

dilation, rotation by \( \frac{k}{|k|} \)

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} : z \mapsto \frac{1}{z}
\]

(inversion)

\[
\frac{1}{z} = \frac{1}{|z|^2}
\]

so geometrically:

We used linear fractional transformations several times in proof of Riemann mapping theorem, particularly to exhibit maps from \( B(0, 1) \) to itself. Let's classify them:

- Takes interior of unit disk to exterior of unit disk.
Proposition: Any conformal map of \( B(0,1) \) onto itself is a linear fractional transformation of the form

\[
T(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z_0} z}
\]

for some fixed \( z_0 \in \mathbb{C} \), \( \theta \in \left[0, 2\pi\right) \). (and all such maps are conformal bijections)

\[\square\]

If: First check \( T \) of this form is conformal bijection of \( B(0,1) \)

If \( |z| = 1 \), then \( |T(z)| = \left| \frac{z - z_0}{1 - \overline{z_0} z} \right| = \frac{|z - z_0|}{|z| |z - \overline{z_0}|} \)

But \( z^{-1} = \overline{z} \) if \( z = 1 \), so \( |T(z)| = \frac{|z - z_0|}{|z - \overline{z_0}|} = 1 \).

\( T \) has singularity at \( z = \overline{z_0}^{-1} \neq D \) if \( z_0 \in D \), so \( T \) an analytic map from \( D \) to \( D \) by maximum modulus principle.

\( T \) invertible with \( T^{-1}(w) = e^{-i\theta} \left[ \frac{w - (-e^{i\theta} z_0)}{1 - (-e^{-i\theta} z_0) w} \right] \)

which is of same form as \( T \), so also maps \( D \) to \( D \). Thus we obtain the conformal bijection.

Now if \( \phi : D \to D \) is any conformal map,

with \( z_0 = \phi^{-1}(0) \), \( \theta = \arg \phi'(z_0) \), then

\( \phi, T \) two conformal maps with \( z_0 \to 0 \)

\( \theta = \arg \phi'(z_0) = \arg T'(z_0) \)

By uniqueness of conformal maps, \( \phi = T \). \( \square \)
Found special class of LFTs mapping $B(0,1)$ to itself, in particular taking unit circle to itself. In general, LFTs map circles and lines to circles and lines. (View result on $\mathbb{C} \cup \{\infty\}$, where lines in $\mathbb{C}$ become circles on $\mathbb{C} \cup \{\infty\}$ passing through $\infty$.)

**Proposition:** Linear fractional transformations take circles in $\mathbb{C} \cup \{\infty\}$ to circles.

**Two proofs:** (I) use decomposition into simpler transformations (Bruhat decomposition).

Clear that translations + dilations map circles/lines $\rightarrow$ circles and lines.

Check for inversion. Circle: $A(x^2+y^2) + Bx + Cy = D$. (*) $A,B,C$ not all 0.

Analyze effect under $z \rightarrow \frac{1}{z}$.

$u := \text{Re}(\frac{1}{z}) = \frac{x}{(x^2+y^2)}$, $v := \text{Im}(\frac{1}{z}) = \frac{-y}{(x^2+y^2)}$

(*) $\Rightarrow +D(u^2+v^2) - Au + Bv = C$ (circle).

(II) Use an invariant: cross-ratio.

To motivate cross ratio, $\text{SL}(2, \mathbb{C})$ is 3 dimensional. $a,b,c,d$ with $ad-bc=1$.

Pick 3 points to determine LFT.

Given any $z_1,z_2,z_3 \in \mathbb{C} \cup \{\infty\}$, map them to $0,1,\infty$ by:

$$T(z) = \frac{(z_2-z_3)(z-\bar{z_3})}{(z_2-z_4)(z-\bar{z_4})}$$

Appropriately interpreted if one of $z_1,z_2,z_3,z_4 = \{\infty\}$.

E.g.: $z_2 = \infty$:

$$T(z) = \frac{z-z_3}{z-z_4}.$$
T is unique since if \( S \) maps \( z_1, z_2, z_3, z_4 \) to \( 1, 0, 0, \infty \) then

\[ S \text{ fixes } 0, 1, \infty. \]

Show only such transformation is identity.

E.g. if \( T(0) = 0 \) then

\[
\begin{align*}
\frac{a(0) + b}{c(0) + d} &= 0 \\
\end{align*}
\]

Similarly:

\[
T(\infty) = 0 \Rightarrow c = 0, \quad T(1) = 1 \Rightarrow a = d.
\]

So define cross-ratio

\[
(z_1, z_2, z_3, z_4) := \frac{z_2 - z_4}{z_2 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_4}
\]

the image of \( z_1 \) under map \( T : (z_1, z_2, z_3, z_4) \) to \( (1, 0, 0, \infty) \).

**Theorem:** \( z_1, z_2, z_3, z_4 \) distinct in \( \mathbb{C} \cup \{0, \infty\} \). \( S \) a LFT, then

\[
(S_{z_1}, S_{z_2}, S_{z_3}, S_{z_4}) = (z_1, z_2, z_3, z_4).
\]

**Proof:**

To \( S^{-1} \) maps \( (S_{z_1}, S_{z_2}, S_{z_3}, S_{z_4}) \) so that

\[
(S_{z_1}, \ldots, S_{z_4}) = T \circ S^{-1}(S_{z_1}) \quad \rightarrow (1, 0, 0, \infty)
\]

Hence

\[
(S_{z_1}, \ldots, S_{z_4}) = T(z_1).
\]

**Corollary:** To map \( z_1, z_2, z_3, z_4 \) to \( w_1, w_2, w_3, w_4 \) by LFT, solve for \( W \) in:

\[
(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4).
\]

To use cross-ratio to prove that LFTs take circles to circles, we show

**Thm:** Cross-ratio \( (z_1, z_2, z_3, z_4) \) is real \( \iff \) \( z_i \)'s lie on circle or line.

**Proof:** Take \( \text{arg} \,(z_1, z_2, z_3, z_4) \) analyze geometrically to show image of \( \mathbb{R} \) is circle or line. (Why is this enough?)
Often enough to combine a few basic transformations with LFTs to get desired conformal maps.

Example: Map upper half plane to unit circle.

Take real axis to unit circle, e.g. \(-1, 0, 1\) to \(i, -1, -i\).

Solve for \(a, b, c, d\) in \(T(z) = \frac{az + b}{cz + d}\) get \(T(z) = \frac{z - i}{z + i}\).

Example 2: Map \(\mathbb{C}\) to unit disk.

Plan: Map half plane at angle \(\theta\) to usual upper half plane by rotation \(z \rightarrow e^{i\theta}z\).

Then compose with Example 1 map. Get \(z \rightarrow \frac{e^{i\theta}z - i}{e^{i\theta}z + i}\).

Other useful basic maps:

- \(z \rightarrow z^2\)
- \(z \rightarrow e^z\)
- \(z \rightarrow \log z\)
- \(z \rightarrow \sin z\)
- \(z \rightarrow \pi/2, \pi/2

\(z \rightarrow \pi/2\)