linear fractional transformations (Ahlfors 3.3)

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{C},\ ad-bc = 1 \right\}$$

matrix group acting on points in $\mathbb{C}$ by, with $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$T \cdot z := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

(just matrix mult: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}$)

which makes clear this is a group action where composition corresponds to matrix multiplication)

Also consider $GL(2,\mathbb{C})$ with $ad-bc \neq 0$.

(note if $ad-bc=0$, then rows are multiples of each other $\Rightarrow$ transform is constant.

These obviously behave differently. want to rule them out.)

Similarly clear that inverse is

$$z \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{dz-b}{-cz+a}.$$

Note $GL(2,\mathbb{C})$ and $SL(2,\mathbb{C})$ differ by matrices of form $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

which is just a scaling $z \mapsto kz$. + rotation

(think $\frac{k}{|k|}$ and ~~angle~~ $\frac{\cancel{\text{angle}}}{|k|}$)

$$\boxed{\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \cdot z \mapsto z \quad \text{so can mod out by this action}}$$

Finally, $T(\infty) := \frac{a}{c}$ useful convention (or extension to Riemann sphere)

$T\left(-\frac{d}{c}\right) := \infty$.

Thus T maps Riemann sphere $\mathbb{C} \cup \{\infty\}$ continuously to itself with continuous inverse. Or $\mathbb{C} \setminus \{-\frac{d}{c}\} \xleftrightarrow[\text{bij.}]{} \mathbb{C} \setminus \{\frac{a}{c}\}$

T, T^{-1} are conformal since $z = T(T^{-1}(z)) = T^{-1}(T(z))$

and so $T'(z)$ non-zero by chain rule.

Every matrix in $GL(2,\mathbb{C})$ can be decomposed as

(Bruhat decomposition) $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix}$ or $\begin{pmatrix} 1 & \alpha_1 \\ & 1 \end{pmatrix}\begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \alpha_2 \\ & 1 \end{pmatrix}$

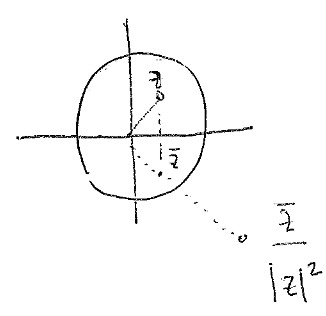
so only need to understand elementary transformations:

$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : z \mapsto z + \alpha$ (translation by vector α)

$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} : z \mapsto kz$ (dilation by $|k|$, rotation by $\frac{k}{|k|}$.)

$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} : z \mapsto \frac{1}{z}$ (inversion)

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ so geometrically:



takes interior of unit disk to exterior of unit disk.

We used linear fractional transformations several times in proof of Riemann mapping theorem, particularly to exhibit maps from $B(0,1)$ to itself. Let's classify them.

Proposition: Any conformal map of $B(0,1)$ onto itself is a linear fractional transformation of the form $T(z) = e^{i\theta} \dfrac{z - z_0}{1 - \bar{z}_0 z}$ for some fixed $z_0 \in B(0,1)$, $\theta \in [0, 2\pi)$. (and all such maps are conformal bijections)

Pf: First check T of this form is conformal bijection of $B(0,1)$

If $|z| = 1$, then $|T(z)| = \left| \dfrac{z - z_0}{1 - \bar{z}_0 z} \right| = \dfrac{|z - z_0|}{|z|\,|z^{-1} - \bar{z}_0|}$

But $z^{-1} = \bar{z}$ if $z = 1$, so $|T(z)| = \dfrac{|z - z_0|}{|\bar{z} - \bar{z}_0|} = 1.$

T has singularity at $z = \bar{z}_0^{-1} \notin D$ if $z_0 \in D$,

so T analytic map from D to D, by maximum modulus principle.

Note T invertible with $T^{-1}(w) = e^{-i\theta} \left[\dfrac{w - (-e^{i\theta} z_0)}{1 - (-e^{-i\theta} \bar{z}_0) w} \right]$

which is of same form as T, so also maps D to D. Thus we obtain the conformal bijection.

Now if $\phi : D \to D$ is any conformal map, with $z_0 = \phi^{-1}(0)$, $\theta = \arg \phi'(z_0)$, then

ϕ, T two conformal maps with $z_0 \mapsto 0$
$\theta = \arg \phi'(z_0) = \arg T'(z_0)$

By uniqueness of conformal maps, $\phi = T$. //

Found special class of LFTs mapping $B(0,1)$ to itself, in particular taking unit circle to itself. In general, LFTs map circles and lines to circles and lines. (View result on $\mathbb{C} \cup \{\infty\}$, where lines in $\mathbb{C}$ become circles on $\mathbb{C} \cup \{\infty\}$ passing through $\{\infty\}$.)

Proposition: Linear fractional transformations take circles in $\mathbb{C} \cup \{\infty\}$ to circles.

Two proofs: (I) use decomposition into simpler transformations (Bruhat decomposition)

clear that translations + dilations map circles/lines $\rightarrow$ circles and lines.

check for inversion. Circle or line: $A(x^2+y^2) + Bx + Cy = D$. $(*)$ A, B, C not all 0.

analyze effect under $z \mapsto 1/z$.

$$u := \mathrm{Re}\left(\frac{1}{z}\right) = \frac{x}{(x^2+y^2)}, \quad v := \mathrm{Im}\left(\frac{1}{z}\right) = \frac{-y}{(x^2+y^2)}$$

$$(*) \Rightarrow +D(u^2+v^2) - Au + Bv = C$$

(circle).

(II) use an invariant: cross-ratio.

To motivate cross ratio, $SL(2,\mathbb{C})$ is 3 dimensional. : a,b,c,d with $ad-bc=1$.

pick 3 points to determine LFT.

Given any $z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$, map them to $1, 0, \infty$ (in this exact order) by:
distinct points

$$T(z) = \left(\frac{z_2 - z_3}{z_2 - z_4}\right) \cdot \frac{z - z_3}{z - z_4}$$

appropriately interpreted if one of $z_2, z_3, z_4 = \{\infty\}$.

e.g. $z_2 = \infty$:

$$T(z) = \frac{z - z_3}{z - z_4}.$$

T is unique since, if S maps z_2, z_3, z_4 to $1, 0, \infty$ then

ST^{-1} fixes $0, 1, \infty$. Show only such transformation is identity.

e.g. if $T(0) = 0$ then $\frac{a(0)+b}{c(0)+d} = 0 \Rightarrow b = 0$

similarly: $T(\infty) = \infty \Rightarrow c = 0$, $T(1) = 1 \rightarrow a = d$.

So define cross-ratio $(z_1, z_2, z_3, z_4) := \frac{z_2 - z_4}{z_2 - z_3} \frac{z_1 - z_3}{z_1 - z_4}$,

the image of z_1 under map $T: (z_2, z_3, z_4)$ to $(1, 0, \infty)$.

Theorem: z_1, z_2, z_3, z_4 distinct in $\mathbb{C} \cup \{\infty\}$. S a LFT, then

$$(Sz_1, Sz_2, Sz_3, Sz_4) = \underbrace{(z_1, z_2, z_3, z_4)}_{T(z_1)}.$$

pf: $T \circ S^{-1}$ maps (Sz_1, Sz_2, Sz_3, Sz_4) so that $(Sz_2, Sz_3, Sz_4) \mapsto (1, 0, \infty)$

hence $(Sz_1, \dots, Sz_4) = T \circ S^{-1}(Sz_1)$

$= T(z_1)$. ✓ //

Corollary: To map z_2, z_3, z_4 to w_2, w_3, w_4 by LFT,

solve for w in: $(z, z_2, z_3, z_4) = (w, w_2, w_3, w_4)$.

To use cross-ratio to prove that LFTs take circles to circles, we show

Thm: Cross-ratio (z_1, z_2, z_3, z_4) is real $\iff$ z_i's lie on circle or line.

2 pfs: take $\arg(z_1, z_2, z_3, z_4)$ analyze geometrically OR

show image of $\mathbb{R}$ is circle or line. (why is this enough?)

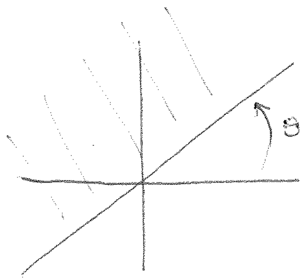
Often enough to combine a few basic transformations with LFTs to get desired conformal maps.

Example: Map upper-half plane to unit ~~circle~~ disk.

Take real axis to unit circle. e.g. $-1, 0, 1$ to $i, -1, -i$.

solve for a, b, c, d in $T(z) = \dfrac{az+b}{cz+d}$ get $T(z) = \dfrac{z-i}{z+i}$.

Example 2: Map



$\rightsquigarrow$ unit disk

Plan: Map half plane at angle θ to usual upper half plane by rotation $z \mapsto e^{-i\theta} z$

then compose with Example 1 map. Get $z \mapsto \dfrac{e^{-i\theta} z - i}{e^{i\theta} z + i}$

Other useful basic maps

$z \mapsto z^2$

$z \mapsto e^z$

$z \mapsto \log z$

$z \mapsto \sin z$