Holonomic modules

**Def.** A left $A_n$-module of degree $n$ is called holonomic. Zero module is holonomic.

$n$ is minimal by Bernstein

**Easy example** $K[X_1, \ldots, X_n]$ is a holonomic module.

$A_n(k) \subset \text{End}_K K[X]$ is not holonomic

For $n=1$, many examples: 

Let $I \neq 0$ be a left ideal of $A_1$.

Recall that $d(A_1/I) \leq 2 \cdot 1 - 1 = 1$

$\Rightarrow d(A_1/I) = 1 \Rightarrow A_1/I$ holonomic
Prop 1
(1) Submodules and quotients of holonomic \(A_n\)-modules are holonomic. \[\exists \text{ exact sig.} \]
(2) \( \mathcal{E}(\text{Holonomic}) \) is holonomic
\[\oplus \text{ Holonomic} \]

Proof
(1) Let \( M \) holonomic \( \otimes \) \( A_n \)-module, \( N \subset M \) submodule. By 9.3.2,
\[ n = d(N) = \max \{ d(N), d(M/N) \} \geq d(N) = n \geq d(M/N) = n \]
By Bernstein, \( N, M/N \) holonomic

(2) Recall by 9.3.3,
\[ d(M, \otimes \cdots \otimes M_k) = \max \{ d(M_i) \} \]
and \( \mathcal{E}(\text{Holonomic}) \oplus \text{ Holonomic} / \]
Corollary: Finitely generated torsion $A_i$-modules are holonomic.

Proof: Let $2u_i, x_i$ be generators of $M$ for $i = 1, 2$. Let $\{b_i\}_{1}^n$ be associated annihilators, i.e.,

$$a_i b_i u_i = 0$$

As thus, $A_i u_i$ is a quotient of

$$A_i u_i = (A_i/A_i b_i) / (a_i)$$

Thus, $A_i u_i$ is holonomic if so is the sum.

Thus for $A_i$-modules, holonomic version.
Observe that if \( I \subseteq J \) are left ideals of \( A_n \), then \( I/J \) is torsion.

\[ R/I \quad J/I \text{ torsion holonomic} \]

Prop 1: Holonomic \( A_n \)-modules are torsion.

Proof 1: Let \( M \) be holonomic \( /A_n \). Let \( 0 \neq u \in M \).

\[ \phi(a) \phi: A_n \to M, \]
\[ a \mapsto au, \]

Clearly \( \phi(A_n) \subseteq M \), so \( d(\phi(A_n)) = n \).

Since \( d(A_n) = \max \{ d(A_n/\ker \phi), d(\ker \phi) \} \)
\[ = \max \{ 2n, 1 \} = 2n. \]

Thus \( \ker \phi \) is non-trivial, so \( u \) is torsion.
For every $M$ fib over $A$,

torsion $\neq$ holonomic

Not true for $n \geq 2$

E.g., $M = An / \mathfrak{a} \mathfrak{m} \mathfrak{a} \mathfrak{m} \mathfrak{a} \mathfrak{m} \mathfrak{n}$

Pf: $E_\infty (9, 5, 2, -3)$
Recall: A module $M$ over ring $R$ is Artinian if all descending strict chains of submodules

$$N_1 \supseteq N_2 \supseteq \cdots$$

is stationary.

Then let $M$ left-module over ring $R$, $N$ submodule of $M$.

1. $M$ Artinian $\iff$ every set of submodules has a minimal element.

2. $M$ Artinian $\iff N$ and $M/N$ Artinian.

3. For $N'$ a submodule, $M=N+N'$

$N', N$ Artinian $\implies M$ Artinian.
pf (1) \( \Rightarrow \) M Artinian, \( S \neq \emptyset \) collection of submodules.

Suppose there is no minimal element.

Choose a submodule \( N \in S \), Not minimal, so \( \exists \) submodule \( N_{n+1} \in S \) s.t. \( N_{n+1} \not\subseteq N_n \), \( \Rightarrow \subseteq \)

Let

\[ N_1 \supseteq N_2 \supseteq \cdots \] be chain of submodules.

Let \( S = \bigcap_{i=1}^{\infty} N_i \), let \( N_n \) be minimal guy.

Obv. stationary.

(2) \( \Rightarrow \) M Artinian, \( N \) submodule.

Any chain of submodules of \( N \) is a chain of submodules of \( M \), so \( N \) Artinian.

Let

\[ 0 \to N \to M \to N'' \to 0 \]

M Artinian \( \Rightarrow \) \( N' \) Artinian.

Let \( \{N_0 \supseteq N_1 \supseteq \cdots \} \) be descending chain.

Let \( M_i = \text{g}^{-1}(N_i) \), so \( M_i \not\subseteq M_{i+1} \).
Conversely, let \( M, N', N'' \) be Artinian.

Let \( \mathfrak{F}_{M} \rightarrow M_{2} \rightarrow \cdots \rightarrow M_{1} \rightarrow M \).

Let \( M_{i}^{n} = g(M_{i}) \), this is stationary,

so \( M_{i} + N = M_{i+1} + N' \) for \( i \geq 0 \),

\( N' \) Artinian \( \Rightarrow \quad f^{-1}(M_{i}) = f^{-1}(M_{i+1}) \) for \( i \geq 0 \),

so \( M_{i} \) must terminate.

(\( \Rightarrow \)) Let \( M = N + N' \), \( N, N' \) Art.

Then \( M/N = (N'/N'\cap N) \) Artinian.

So \( M \) is.
Theorem: Holonomic Modules are Artinian

Proposition:

Let \( M = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_r \)

such that all have \( d(i) = n \)

Then

\[
\begin{align*}
  m(N_i) &= m(N_{i+1}) + m(N_{i+1}/N_i) \\
  \text{Thus } m(M) &= \sum_{i=0}^{r-1} m(N_i/N_{i+1}) + m(N_r) \\&\geq r
\end{align*}
\]

Not all \( \mathfrak{m} \)-modules are Artinian

E.g., \( \mathfrak{m} \) over itself

\[
\mathfrak{m} \times \mathfrak{m} \supseteq \mathfrak{m} \times \mathfrak{m}^2 \supseteq \mathfrak{m} \times \mathfrak{m}^3 \supseteq \cdots
\]
Consider that $N_0 \leq N_1 \leq N_2 \leq \ldots \leq N_k$.

Let $M$ be a left module over $R$.

- Artinian
- Noetherian

Let $\mathfrak{a}(N_{i+1}/N_i)$ be simple.

Consider $S = \{ \text{submodules of } \frac{M}{N_k} \}$, Artinian.

Let $N_k+1$ be minimal element. Then we obtain

$N_0 \leq \ldots \leq N_k \leq N_{k+1}$.

Noetherian $\Rightarrow$ terminates.

$O = N_0 \leq N_1 \leq \ldots \leq N_k = M$ called composition series.

Fact (Artin-Schreier) well-defined.

The # of gaps is called the length of $M$.
Scholium: \( L(M) \leq m(M) \) for \( M \) holonomic.

Proof:

1. \( 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M \)

Have:

1. Each \( M_i \) holonomic
2. \( M_i/M_{i-1} \) holonomic

\[ \therefore m(M) = \sum_{i=1}^{l} m(M_i/M_{i-1}) \geq l \]

Corollary: Let \( M \) be \( R \)-module. \( M \) is holonomic if \( m(M) = 1 \)

\[ \Rightarrow M \text{ simple} \]

Proof:

Suppose \( N \subset M \). Then \( N, M/N \) holonomic.

So

\[ m(M) = m(N) + m(N) = m(M/N) = 1 \]

\[ \Rightarrow M/N = 0 \]

\[ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 \\ \end{bmatrix} \Rightarrow M = N \]

So \( M \) is simple.
Thm] Let \( R \) be simple left Artinian and \( M \) be a \( R \)-module. Then

\( (M \text{ Artinian}, R \text{ not } \mathbb{R}\text{-Artinian}) \implies M \text{ cyclic} \)

**Proof:** Observe that \( l(M) \) is well defined.

**Idea:** If \( M = Ru + Rv \), we want to show that \( \text{Frac} R \subset \begin{array}{c} M \cong R(u + av) \end{array} \).

Induct on length: if \( l(M) = 1 \), then \( M \text{ irr} = \begin{array}{c} \text{for } a \neq 0 \end{array} \).

Suppose the statement holds for length \( < r \) and \( M = Ru + Rv \) with \( l(M) = r \),

\[ \implies l(Ru) \text{ finite} \implies \exists \text{ for some } v \in R \text{ such that } Rv \text{ irr} \subset Rv \).

Thus \( l(M/Rv) < r \), so (by induction hypothesis)

\[ M = R(u + xv) + Rv \text{ for some } x \in R. \]

Rewriting, \( M = Ru + Rv \text{ irr} \).
Let $\phi : R \to M$

Note: $\phi$ cannot be injective, since $(\inf)_{\text{submodule}} \subseteq M$ must be Artinian.

Thus $\exists 0 \neq d \in \text{Ker } \phi$.

Claim: $R(u + dv) = R$.

$R$ simple $\Rightarrow RdR = R$ $\Rightarrow RdRv \neq 0$.

Thus $\exists b \in R$ s.t. $dbv \neq 0$.

Claim: $M = R(u + dv)$.

$d(u + dv) = dbv \neq 0$.

Thus $Rdbv = Rv = \Rightarrow v \in R(u + dv)$

$\Rightarrow v \in R(u + dv)$

Cor. Holonomic modules are cyclic.

Proof: $M$ holonomic must be Artinian, $A_0$ not Artinian.
Lemma: Let $M$ be a left $A_n$-module with a filtration $\mathcal{F}$ wrt. Bernstein filtration of $A_n$. Suppose $c_1, c_2$ constants s.t.

$$\dim_k \mathcal{F}_j \leq \frac{c_1 j^n}{n!} + \frac{c_2 (j+1)^{n-1}}{n!} \quad (j \geq 0)$$

Then $M$ is holonomic $\mathcal{D}$ of multiplicity $\leq c_1$.

Proof: Claim 1: Every $S_j$ submodule of $M$ satisfies theorem.

Proof of claim:

Let $N \leq M$ be $S_j$. $N$ admits a good Hilbert polynomial $X(t)$.

By 8.32, there exists $r \in \mathbb{Z}^+$ s.t. $\mathcal{N}_r \leq \mathcal{F}_j \cap N$.

Then

$$\dim_k \mathcal{N}_r \leq \dim_k \mathcal{F}_j$$

$$\chi(j) = \dim_k \mathcal{N}_r \leq \dim_k \mathcal{F}_j$$

$$\leq \frac{c_1 (j+1)^n}{n!} + \frac{c_2 (j+1)^{n-1}}{n!} \quad (j \geq 0)$$

$$\Rightarrow \deg X(t) \leq n,$$ 

Bernstein $\Rightarrow \deg \chi(t) = \deg \chi(t) = 7m(N) \leq c_1.$
Now consider ascending chain

\[ N_1 \subseteq N_2 \subseteq \ldots \subseteq N_r \]

\[ \text{e.g., submodules.} \]

\[ \text{holonomic } \dim \eta_j \]

\[ \Rightarrow m(N_i) = m(N_{i-1}) + m(N_i/N_{i-1}) \]

\[ \Rightarrow \sum_{i=1}^r \frac{m(N_i/N_{i-1}) + m(N_1)}{2} = m(N_r) \]

\[ \leq c_1 \]

\[ \Rightarrow \text{all ascending chains of } \text{e.g., submodules have length } \leq c_1. \]

\[ \Rightarrow M \text{ is f.g.}. \]
Let $k(X) = k(x_1, \ldots, x_n)$ be the field of rational functions. The left action of $A_n$ on $k[X]$ extends naturally, with $x_i$ acting by mult, and

$$
\partial_i \cdot \frac{f}{g} \quad \text{by quotient rule}
$$

$$
\partial_i \left( \frac{f}{g} \right) = \frac{\partial_i (f) g - f \partial_i (g)}{g^2}
$$

Fixing some nonzero polynomial $p \in k[X]$, one can consider

$$
k[X, p^{-1}] = \left\{ \frac{f}{p^r} \mid f \in k[X] \right\}
$$

Since partial derivatives of such $f$'s have denominator $p^{2r}$, $k[X, p^{-1}]$ is a left $A_n$-submodule of $k(X)$. 
Thm: \( M = k[x_1, \ldots, x_n] \) is holonomic with \( m(M) \leq (\deg p + 1)^n \).

Proof:

Let \( n = \deg p \),

\[
\Pi_k = \left\langle \frac{f}{\rho^k} \mid \deg(f) \leq (m+1)k \right\rangle
\]

Claim: \( \Pi \) is a filtration of \( M \).

1. Let \( \frac{f}{\rho^k} \in \Pi \), \( \deg f = s \). Then

\[
\frac{f}{\rho^k} = \frac{f \cdot \rho^s}{\rho^{s+k}}.
\]

Now \( \deg(f \cdot \rho^s) = s + m = s(m+1) \leq (m+1)(s+k) \)

\[\Rightarrow \frac{f}{\rho^k} \in \Pi_{s+k}, \Rightarrow M = \bigcup_{k=0}^{\infty} \Pi_k. \checkmark\]

2. Let \( \frac{f}{\rho^k} \in \Pi_k \), \( \deg f \leq (m+1)k \)

\[\Rightarrow x_i(f/\rho^k) = \frac{x_i f \rho^k}{\rho^{k+1}} \in \Pi_{k+1} \]

and

\[
\partial_e x_i \left( \frac{f}{\rho^k} \right) = \frac{p^k \partial_i(f) - f \cdot \partial_i(p^k)}{p^{2k}}
\]

\[= \frac{p^k \partial_i(f) - f \cdot K \partial_i(p)}{p^{k+1}}.\]
When numerator has \( \deg \leq (m+1)k + (m-1) \)

\[ \varepsilon^{(m+1)}(k+1) \]

\[ \Rightarrow \exists i \in \left( \frac{k}{m} \right) \in \Gamma_{k+1}. \]

Thus \( B_1 \cdot \Gamma_k \subseteq \Gamma_{k+1} \Rightarrow B \cdot \Gamma_k \subseteq \Gamma_{i+k} \)

\[ \Rightarrow \dim (\Gamma_k) \leq \dim \text{ vect. space of poly. of deg } \leq (m+1)k \]

\[ \Rightarrow \Gamma_k \text{ is f.d.} \]

So \( \bigotimes \Gamma \) is a filtrable and

\[ \dim_n \Gamma_n \leq \binom{(m+1)k + n}{n} \]

Calculation shows:

\[ \dim_k \Gamma_k \leq \frac{(m+1)^n k^n}{n!} + \frac{(m+1)^{n-1} (n+1) n (k+1)^{n-1}}{n!} \]

for \( k \gg 0 \)

\[ \Rightarrow \text{ by Lemma 3.1, that} \]

\( M \) is holonomic, \( m(M) \leq (m+1) \). \( \square \)
Bernstein polynomial

Fix $p \in K[X]$. Let $s$ be a variable, $K(s)$ field of rational functions.
Let $A_n(s)$ be the Weyl algebra over $K(s)$.
Let $A_n(K(s))p^s$ denote the $A_n(k(s))$-module generated by $p^s$, a formal symbol, $s,t$.

\[ \partial_s p^s = s p^{-1} \partial_s (p) \cdot p^s. \]

Let $\Delta$ be an automorphism of

\[ A_n(K(s))p^s \subset K(s)[X,p^{-1}]p^s, \]

Let $+$ be the auto of $K(s)[X,p^{-1}]p^s$ s.t.
\[ + (s^i p^s) = (s+1)^i p \cdot p^s. \]

N.B. $A_n(k)$-linear, not $A_n(K(s))$-lin.
Thm 2.1.2

There exist $B(s) \in K[s]$ and a differential operator $D(s)$ in $A_n(K)[s]$ such that

$B(s) \rho^s = D(s) \rho \rho^s$

Proof: Fact: $K(s)[X, \rho^{-1}] \rho^s$ is holonomic. Analogous to proof that $K[X, \rho^{-1}]$ is holonomic.

Thus $A_n(s)\rho^s$ is holonomic. Thus it has finite length, and

$A_n(\ell s)\rho^s \supset A_n(s)\rho\rho^s \supset A_n(s)\rho^2\rho^s \supset \ldots$

is stationary.

So $\exists k > 0$ s.t.

$\rho^k \rho^s \in A_n(s)\rho^{k+1}\rho^s$

$\Rightarrow$ (after applying $t^{-k}$) that $\rho^s \in A_n(s)\rho\rho^s$.

We can clear denominators and obtain that $\exists B(s) \in K[s]$ such that

$B(s)\rho^s \in A_n(K)[s]\rho\rho^s$. 

End
Notes: \((B(s), D(s))\) not uniquely determined, but set of \(B(s)\) is an ideal and has a unique monic generator, called the Bernstein polynomials.

Hard to calculate generally.