\( K := \frac{A}{\mathfrak{m}}: \text{max. ideal} \) (with notation as in Proposition 3)

\( \pi: A \rightarrow \bar{A}: \text{can. projection.} \)

\( q: \bar{A} \rightarrow K \)

\[ 0 < s := \text{length} \bar{M} := \frac{M}{\mu M} \mu: \text{central} \]

For any \( a \in A \), \( \rho(a): M \rightarrow M \quad \rho(a)(x) = ax \), \( \rho(a) \in \text{End}_A(M) \)

and \( \rho(a) \in \text{End}_A(M) \) if \( a \) is central. In particular \( \rho(\mu) \in \text{End}_A(M) \),

and \( \ker \rho(\mu) = \text{Im} \rho(\mu) \).

\[ \begin{align*}
\text{Rank: } 0 & \hookrightarrow \mu M \twoheadrightarrow M \twoheadrightarrow \bar{M} \cong 0 + \bar{M} \\
& \mu M & \text{ finite length} \iff M \text{ finite length} \end{align*} \]

Set \( J := \mu A \). \quad [J, A] := [\mu A, A] = \mu [A, A] = \mu^2 A = 0 \)

so \( J \in \mathcal{Z}(A) \): center

For every \( a, b \in A \), \( [a, b] \in J \), hence \( \rho([a, b]) \in \text{End}_A(M) \).

Put \( [a, b] = \mu c \) so that \( \{ a, b \}^\mu = \bar{c} \).

\[ \rho([a, b])(M) = \rho(\mu)(M) = \mu M. \quad \text{So we have } \bar{\rho}(\bar{[a, b]}) \in \text{End}_A \bar{M} \]

which is just mult. by \( \bar{c} \).

Now \( \bar{M}^s = 0 \). For each \( i = 0, \ldots, s-1 \), we

have the \( K \)-vector space \( \bar{M}^i / \bar{M}^{i+1} \)

with

\[ \sum_{i=0}^{s-1} \dim_K \frac{\bar{M}^i}{\bar{M}^{i+1}} = s. \quad W_i \]

Let \( \bar{\rho}([a, b])_i \) be the induced \( K \)-linear map on \( W_i \), \( i = 0, \ldots, s-1 \) which is just mult. by \( g(\bar{c}) \in K \).

\[ \text{tr}'(\bar{\rho}([a, b])) := \sum_i \text{tr}_K(\bar{\rho}([a, b])_i) = s g(\bar{c}). \quad \text{So } tr' = 0 \]

only if \( g(\bar{c}) = 0 \).
We need \( q(\tau) = 0 \) if \( \tau \in \mathfrak{g} \). Hence we are finished if we show:

if \( a, b \in \pi^{-1}(\mathfrak{g}) \subset A \), then \( \text{tr}'(p([a, b])) = 0 \).

**Key point:** if \( a \in \pi^{-1}(\mathfrak{g}) \), then \( p(a) \in \text{End}_{\pi^*(\mathfrak{g})}(M) \) is nilpotent.

Since \( \pi^*(\mathfrak{g}) = 0 \), and so \( \pi^*(\mathfrak{g})^2 M \subset \mu M \), hence \( \pi^*(\mathfrak{g})^2 M \subset p(\mu)^2 M = 0 \).

Thus works for all elts of \( \pi^{-1}(\mathfrak{g}) \).

We rephrase the problem: \( A \): ring, \( I \subset A \) two-sided ideal, \( I \subset \pi(A) \), and such that \( A/I \) is comm. local ring, \( \pi : A \to \pi(A) \) maximal ideal, \( K = A/\pi(A) \) with \( \text{char}(K) = 0 \).

\( M \): finite length \( A \)-module.

Two endomorphisms \( T_1 = p(a) \), \( T_2 = p(b) \in \text{End}_{\pi}(M) \).

\( \mu = p(\mu) \in \text{End}_A(M) \) satisfying:

1. \( \text{Im } \mu = \text{Ker } \mu \) and \( \text{Im } \mu = \mu M \)
2. \( [T_i, p(A)] \in p(J) \) and \( [T_i, p(J)] = 0 \) for \( i = 1, 2 \)
   \( (p(A) \subset \text{mult. by elts in } A, \text{ as elts of } \text{End}_{\pi}(M)) \)
3. \( [T_i, \mu] = 0 \) \( (i = 1, 2) \) and \( [T_1, T_2](M) \subset \mu M \). \( (\text{Let } T_0 := [T_1, T_2] \text{ which is } A \text{-linear}) \)

and induces \( \widetilde{T}_0 \in \text{End}_{A}(\widetilde{M}) \) where \( \widetilde{A} := A/I \), \( \widetilde{M} := M/\mu M \).

\( \widetilde{T}_1, \widetilde{T}_2 \) nilpotent.

Want: \( \text{tr}'(\widetilde{T}_0) \) (defined as before via successive quotients) = 0.

By replacing \( A \) with \( A/\pi^{-1}(\mathfrak{g})^{2s} \) and \( J \) by \( J + \pi^{-1}(\mathfrak{g})^{2s} \), we may assume \( \mu \) is nilpotent. \( \Rightarrow \widetilde{A} \) is complete local ring of equal char.
\[ \exists \text{ subfield } F \subset A/\mathfrak{J} \text{ complementary to } \mathfrak{m} \text{ where} \]

\[ q : A/\mathfrak{J} \longrightarrow (A/\mathfrak{J})/\mathfrak{m} = K \text{ induces an isom. } q : F \longrightarrow K \]

and thus \( \text{tr}'(\overline{T}_0) = q(\text{tr}_F(\overline{T}_0)) \).

Let \( B := \pi^{-1}(F) \subset A \) (subring). \( \mathfrak{J} \subset B \) central two-sided ideal.

\( M \) has finite length as \( B \)-module.

All data \( \mu, T_1, T_2 \) satisfies some properties 1-4 above relative to \( B \).

So replacing \( A \) by \( B \) in all instances above, we may assume \( A/\mathfrak{J} = K \) is a field.

(i.e. regard \( \overline{T}_0 \in \text{End}_K(\overline{M}) \). So can use linear algebra / choice of basis to compute trace).

For \( x \in M \) let \( \tau : M \rightarrow \overline{M} = M/\mu M \). We assumed \( T_0(M) \subset \mu M \) so

\[ x \mapsto \tau(x) = \overline{x} \]

Let \( \xi e_1, \ldots, \xi e_\ell \subset M \) be s.t. \( \xi \overline{e}_1, \ldots, \xi \overline{e}_\ell \) is a \( K \)-basis of \( \overline{M} \).

\[ T_0 e_i = \mu \sum_j g_{ij} \xi e_j, \text{ so that } \overline{T}_0 \overline{e}_i = \sum_j \pi(g_{ij}) \overline{e}_j \]

then \( \text{tr}(\overline{T}_0) = \pi(\text{tr} G) \text{ G = (g_{ij})} \)

Key Lemma: One can choose \( e_1, \ldots, e_\ell \) s.t. \( G = Z + F \) with \( Z \) upper triangular.

\[ F \text{: sum of two commuting matrices.} \]