§1: D-modules with support

for a module $M$ over a commutative ring $R$, and an ideal $I \subseteq R$

we may define

$$\Gamma_{V(I)}(M) := \left\{ u \in M \; s.t. \; \exists n \geq 0 \; I^n u = 0 \right\}$$

For our set-up, we have:

$$X \xrightarrow{\nu} X \times Y \quad \nu(x) = H$$

For an $A_{ntm}$-module $M$, define $\Gamma_H(M)$ by viewing $M$ as $K[X \times Y]$-module (or as $K[Y]$-mod.)

Lemma: $\Gamma_H(M) \subseteq M$ is an $A_{ntm}$-submodule

pf: All clear except for action by $d_{y_j}$.

If $u \in \Gamma_H(M)$, $y^k_j u = 0$ (or more generally $y^k u = 0$) then $y^{d+e_j} d_{y_j} u = 0$

so $d_{y_j} u \in \Gamma_H(M)$.

If $\Gamma_H(M) = M$ then we say $M$ has support on $H$.

Given any $A_{ntm}$-module, $\ker_M(y) = \left\{ u \in M : (y) u = 0 \right\}$

this is only $A_n$-module

not an $A_{ntm}$-module.

Let $M_0 := \ker_M(y)$. Then $A_{ntm}(M_0) = \Gamma_H(M)$. 
Lemma: Suppose $m=1$, then as

\[ \text{Anti}-\text{modules, we have (for } i : x \rightarrow x \times y) \]

\[ i_\ast (M_0) \cong M_0 \otimes K \cong \text{Anti} M_0 = \Gamma_H(M) \]

We must show $\text{Anti} M_0 = \Gamma_H(M)$.

Suppose $u \in \Gamma_H(M)$ and $y^d u = 0$. Want $u \in \text{Anti} M_0$.

For $\alpha = 1$, $y u = 0 \Rightarrow u \in M_0$ by definition.

Assume $\alpha > 1$, true for all smaller $\alpha$ by induction.

\[ 0 = \partial y^d u = y^d \partial y u + \alpha y^{d-1} u = y^{d-1} (\partial y u + \alpha u) \]

\[ \Rightarrow y \partial y u + \alpha u \in \text{Anti} M_0 \text{ by induction.} \]
On other hand, \( 0 = y^d u = y^{d-1}(y u) \Rightarrow y u \in \text{Ann} M_0 \)

\[ \Rightarrow \delta y y u \in \text{Ann} M_0 \]

\[ \Rightarrow \delta u + y \delta y u - \delta y y u = (d-1) u \in \text{Ann} M_0. \]

\[ \text{§2. We can reformulate the last lemma as} \]

**Thm.** (Kashiwara's equivalence)

\[ \nu : X \rightarrow \nu x \nu Y \quad (Y = A^m_k) \]

\[ \nu x \rightarrow (x, 0) \]

then \( \nu_* \) induces an equivalence of categories \( \mathcal{M}^n = (A^n\text{-mod}) \)

and \( \mathcal{M}^{n+m}(H) = \left( \text{A}^{n+m}\text{-mod}\text{ with support on } H : \nu(x) \right) \)

\[ \nu^n(X) \longrightarrow \mathcal{M}^{n+m}(H) \]

\[ N \longrightarrow \nu_\ast N \]

\[ M_0 = \text{Ker}_M(y) \longleftrightarrow M \]

\[ \text{if: When } m > 1, \quad \nu : X \rightarrow \nu x \nu Y \text{ can be factored as} \]

\[ X \xleftarrow{\nu_1} X \times Z \xrightarrow{\nu_2} (X \times Z) \times W \]

\[ \text{A}^{n-1}_k \quad \text{A}^n_k \]

First \( \nu_\ast N = (\nu_2)_\ast (\nu_1)_\ast N \). Moreover

\[ \text{Ker} (\text{Ker}_M \nu_2(X \times Z)(\nu_1(X))) = \{ u \in M : y_1 u = 0, \ldots, y_m u = 0 \} = \text{Ker}_M(\nu(x)) \]

So we have reduced to case \( m = 1 \). By previous lemma, \( \nu_\ast M_0 \sim \text{Ann} M \)

\[ \Gamma^\ast_H(M) = M. \]
If $M \rightarrow M'$ is a morphism of $A_{n+1}$-modules, this induces a map $M_0 \rightarrow M'_0$, which induces $M = A_{n+1} M_0 \rightarrow A_{n+1} M'_0 = M'$. On the other hand, since only elts of $K[\partial y]$ annihilated by $y$ are the constants in $K$, we have:

$$\text{Ker } \psi_*(M_0) \cong \text{Ker } M_0 \otimes K[\partial y]$$  

If $M_0 \rightarrow M'_0$, then $\psi_*(M_0) \cong M_0 \otimes K[\partial y] \rightarrow \psi_*(M'_0) = M'_0 \otimes K[\partial y]$ induces the same morphism $M_0 \rightarrow M'_0$.

§3 If $M$ holomorphic, then so is $t^*M$. (*)

By induction, we may assume $m = 1$.

Lemma 1: If $M$ is $A_{n+1}$-module and $M' = M / \Gamma_H(M)$, $H = t^* \mathfrak{g}[x]$, then $t^*(M) = t^*(M')$.

If: $b/c \quad t^*M = M / yM$; but we know $y \cdot \Gamma_H(M) = \Gamma_H(M)$. \hspace{1cm} \hspace{1cm} \hspace{1cm} y(A_{n+1} M_0) = A_{n+1} M_0$.

Lemma 2: (*) is true if $\Gamma_H(M) = \mathfrak{g}[x]$.

If: Take $\Gamma$: any good filtration on $M$. (for Bernstein filtration on $A_{n+1}$) Take $\Omega_j = (\Gamma_j + yM) / yM$ induced by $\Gamma_j$ (for $A_n$).

Since $\Omega_j \cong \Gamma_j / \Gamma_{j-1} yM$ and $y \Gamma_{j+1} = \Gamma_j yM \Rightarrow \dim_k(\Omega_j)$ $\leq \dim_k(\Gamma_j)$.
Since $\Gamma_i(M) = 0$, left multiplication by $y$ is injective

\[ \Rightarrow \dim_k(y\Gamma_j) = \dim_k(\Gamma_{j-1}) \Rightarrow \dim_k(\Omega_j) \leq \dim(\Gamma_j) - \dim(\Gamma_{j-1}) \]

1. For $j > 0$, $\dim(\Omega_j) \leq \chi(j, M, \Gamma)

\begin{align*}
\frac{r(m(M))}{n!} \cdot \frac{n}{n} & \quad \text{(via holonomy of $M$ here)} \\
\text{same leading term} & \\
\Rightarrow c \in \mathbb{C} \\
\dim_k(\Omega_j) & \leq \frac{r(m(M))}{n!} \cdot j^{n-1} + c \cdot (j+1)^{n-2}
\end{align*}

2. Lemma 10.3.1 says that $M/yM$ is also holonomic (as $A_m$-module)

with $r(M/yM) \leq r(M)$. / \[ \]

If $F: X \to Y$ polynomial, then $F^*(\text{holon.})$ is holom.

$F_{\#}(\text{holon.})$ is holom.

via factorization of $F$ into $\tau, G, \pi$.

Remark: $\pi: X \times Y \to Y$. We want $\pi_\# N$ to be holonomic if $N$ holon.

$\pi_\# N = N/(d_x)N$ is the Fourier transform of

\[(\tau')^* N' \simeq N'/(x_i)N' \quad \text{where $N'$ = Fourier transform of $N$} \]

$\tau': Y \to X \times Y$

$y \mapsto (x, y)$