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push the glass.

\[ R = k[x_1, \ldots, x_n] \quad k: \text{field.} \]

\[ d_i = \frac{\partial}{\partial x_i}, \quad R \to R. \]

If \( \text{char}(k) = p > 0 \), then \( d_i^p = 0 \). (easy check)

\( \text{Blanket assumption: } \text{char } k = 0 \) \( (\text{so } d_i^t \neq 0, \forall t) \)

\( X_i: R \to R \), (multiplied by \( x_i \)).

\( \text{Def: } A_n(k) \text{ is the } k\text{-subalgebra of } \text{End}_k(R) \)

\( \text{generated by } \{ x_1 \sim x_n, d_i \sim d_n \}. \)

\( A_n(k) \text{ is } \text{the Weyl algebra}. \)

**Relations**

1. \[ [x_i, x_j] = 0. \]
2. \[ [d_i, d_j] = 0. \]
3. \[ [d_i, x_j] = 0 \quad \text{if } i \neq j. \]
4. \[ [d_i, x_i] = 1. \]
Prop: monomials \( \{ x_1^{i_1} \ldots x_n^{i_n} d_1^{s_1} \ldots d_n^{s_n} \} \)

are a \( k \)-basis of \( \operatorname{An}(k) \).

(The relations imply that \( f \) is the generators. For linear independence, consider the degrees of \( x_i \)'s when it acts on \( x_1^{c_1} \ldots x_n^{c_n} \). Consider the action of the highest degree elements of the relation.

Pick a new set of generators of \( R \):

\[ y_1, \ldots, y_n, \quad y_i = f_i(x_1, \ldots, x_n) \quad \text{s.t.} \]

\[ R[x_1, \ldots, x_n] = R[y_1, \ldots, y_n] = R \]

Let \( \overline{d_i} = \frac{\partial}{\partial y_i} : R \to R \) (w.r.t. \( y_1, \ldots, y_n \)).

Then: \( \alpha : \operatorname{An}(k) \to \operatorname{An}(k) \) sending \( x_i \to y_i \),
\( \overline{d_i} \to \overline{d_i} \)

is an automorphism of \( \operatorname{An}(k) \).

Degree \( \delta \) of \( \varphi \in \operatorname{An}(k) \) is \( \max(\Sigma i_i \sigma_i + \Sigma j_i \sigma_i) \).

Fact: \( (x_1^{i_1} \ldots x_n^{i_n} d_1^{s_1} \ldots d_n^{s_n}) (x_1^{i_1'} d_1^{s_1'} \ldots d_n^{s_n'}) \) has leading term \( x_1^{i_1 + i_1'} d_1^{s_1 + s_1'} \).
with the rest of terms' degree at most (top - 2).

Thm: ① $\deg(s + s') \leq \max\{\deg s, \deg s'\}$

② $\deg(s_1 \cdot s_2) = \deg s_1 + \deg s_2$. (need some check)

③ $\deg [s, s'] \leq \deg s + \deg s' - 2$.

Corollary: $An(k)$ is a domain.

Thm: $An(k)$ is a simple ring.

(no nontrivial two-side ideals)

Proof: A $s \in An(k)$, multiplying on left or right, you will get the whole ring.

Ex: $[d_i^t, x_i] = t \cdot d_i^{t-1}$, $[d_i, x_i^t] = t \cdot x_i^{t-1}$.

Then, we use the induction on $\deg(s)$.

Consider $[s, x_i]$ or $[s, d_i]$. Some discussions...
Corollary: Every endomorphism of $A_n(k)$ is injective.

Fact: Every left ideal is generated by 2 elements.

Assume $\text{char}(k) = p > 0$.

Fact: Set $R_n = k[x_1, \ldots, x_n, d_1, \ldots d_n]$ subject to relations $(d_i$ is not differentials).

$R_n$ is not simple.

Example: $R_1 = k[x_1, d_1]$

Claim: $[d_1, x_1^p] = 0$.

So $(x_1^p)$ is a 2-side ideal.

$A_n(k) = R \langle \frac{1}{t!} \frac{d}{dx_1^t} \rangle$ is also considered.