General definition of differential operators:

$K$: commutative ring,

$R$: commutative $K$-algebra.

$D_k(R)$: the ring of $K$-linear Operators.

$S: R \rightarrow R$

$D_k(R)$ is a subring of $\text{End}_K(R)$.

$D_k(R) = \bigcup_{n=0}^{\infty} D_k^n(R)$.

$D_k^n(R)$ is the left $R$-module of $K$-linear diff. operators of order $\leq n$.

Def: $\delta \in D_k^n(R)$ iff $\delta$ is the multiplication by an element of $R$.

$\delta \in D_k^n(R)$ if it is $K$-linear and $[\delta, a] \in D_k^{n-1}(R)$ for any $a \in R$. 
Corollary.

$D_k^n(R)$ is a left $R$-module.

**Proof.** Let $r \in R$, $\delta \in D_k^n(R)$, and $a \in R$, then,

$$[r\delta, a] = r [\delta, a] + [r, a] \delta = r [\delta, a] \in D_k^{n-1}(R)$$

by the inductive assumption on $n-1$.

* (So of course it is also a right $R$-module)

**Proposition:** $D_k(R)$ is a ring.

($D_k(R) \subset \text{End}_k(R)$, so multi. is already defined).

More precisely, $\delta' \in D_k^m(R)$, $\delta \in D_k^n(R)$, then $\delta \cdot \delta' \in D_k^{m+n}(R)$.

**Proof:** Induction on $m+n$.

$$[\delta \delta', a] = [\delta', [\delta, a]] + [\delta, a] \delta'$$

*(when $m=0$ or $n=0$, reduced to Module case)*

*Up to now, nothing tells you there exists a diff operator. Alg. def. is always like this.*
Recall: A \( K \)-linear derivation is a \( K \)-linear map \( \delta : R \to R \), s.t.
\[ \delta(ab) = a \delta(b) + \delta(a) b. \] (e.g. \([- , c]\))

\( \text{Der}_K(R) \) is naturally a left \( R \)-module.

(Messing: Also a \( K \)-lie alg.)

Lemma: \( \text{Der}_K(R) \oplus R = D^1_K(R) \)

**Proof:** \( \delta \in \text{Der}_K(R) \)

What we need: \([\delta, a] \in D^0_K(R)\).

\[ [\delta, a] : r = \delta(ar) - a \delta(r) = \delta(a) : r \] This proves \( \subseteq \)

(1) If \( \delta \in D^1_K(R) \), set \( \delta' = \delta - \delta(1) \)

*(Taking action on \( i \) gives a projection*)

By def. of \( D^1_K \), \([\delta', a], b] = 0, \forall a, b \in R \).

Apply to \( i \), we have:
\[ \delta'(ab) = a \delta'(b) + b \delta'(a) \]
\[ \delta'(ab) - a \delta'(b) - b \delta'(a) = 0. \] *very good!"
Prop: Every \( K \)-linear derivation of \( R = K[x_1, \ldots, x_n] \) is of the form \( \delta = \sum f_i \frac{\partial}{\partial x_i} \). \( \delta \in \text{Der}_K(K[x_1, \ldots, x_n]) \), then

\[
\delta(x_1^{s_1} \cdots x_n^{s_n}) = \sum \frac{s_i}{s_i!} \delta(x_i)^{s_i} x_1^{s_1-1} \cdots x_n^{s_n-1}
\]

\[
\delta x_i = s_i x_i^{s_i-1} \delta(x_i) = \delta(x_i) \cdot \frac{\partial}{\partial x_i}(x_i^{s_i}).
\]

\( \delta(x_i) = f_i \) is a good way to find \( f_i \).

(\( \text{This method can apply to higher degrees.} \))

Thm: \( \text{An}(K) \) is the ring of \( K \)-linear diff. operators of \( R \) (provided \( K \) contains a field of char 0).
Exer: \( R = k[x, t^2] \).

\( \text{Dk}(R) \) is not generated by \( \text{Der}_k(R) \).

**Fact:** If \( R \) is the coordinate ring of a non-singular variety over \( k \) with char 0, then \( \text{Dk}(R) \) ARE generated by \( \text{Der}_k(R) \).

**Fact:** \( \text{D}_k(k[x_1, \ldots, x_n]) \) is generated by

\[ \left\{ \frac{1}{t!} \frac{d^t}{dx_i^t} \right\} \quad \forall \ K \end{equation} \]

(e.g. \( K = \mathbb{Z} \), but not \( \mathbb{F}_p \)).

Same for \( k[[x_1, \ldots, x_n]] \). (\( D_k \rightarrow \text{continuous} \) in this case)