The Bernstein filtration on $A_n(k)$.

- Given a finitely generated left $A_n(k)$-module, we are going to define a dimension $d(M)$, then we will define and study the class of holonomic modules.

$A_n(k)$, Bernstein filtration $\to B$. $\text{gr}^B(A_n(k))$ associated graded alg.

$M$, $\Gamma$, filtration $\to \text{gr}^T(M)$, graded module over $\text{gr}^B(A_n(k))$.

$\text{gr}^B(A_n(k))$ is commutative, has 2-side ideals!

Apply Comm. Alg. to define dimension.
Definition.
A graded ring (resp. k-alg.) is a ring (resp. k-alg.) $R$ that admits a decomposition

$$R = \bigoplus_{i \geq 0} R_i$$

called $i$th homogeneous component.

as Abelian groups (or $k$-vector spaces), s.t.

$$R_i \cdot R_j \subset R_{i+j} \quad \forall i, j \geq 0$$

Polynomials $R[x_1, \ldots, x_n]$ is of course graded, but the grade may not be degree!

Let $R$ be a graded ring, $M$ is an $R$-module. If

$$M = \bigoplus_{i \geq 0} M_i$$

(as Abelian group) s.t.

$$R_i M_j \subset M_{i+j}, \forall i, j \geq 0.$$
How do we give a "grade" on the $\text{An}(k)$?

The degree doesn't work because the relation is not homogeneous.

- Filtrations (to solve the problem above!)
  
  R: $k$-alg., $F$: filtration on $R$, is:

  $F_0 \subset F_i \subset C \ldots$

  All are $k$-subspaces of $R$, s.t.

  1. $R = \bigcup_{i \geq 0} F_i$ and
  2. $F_i F_j \subset F_{i+j}$, $\forall i, j \geq 0$.

  Two filtrations on $\text{An}(k)$:

  1. Bernstein filtration $\mathcal{B} = \{ B_\leq i \}_{i \geq 0}$ by degree $\leq i$.
  2. Order filtration $C_i = \left\{ \text{Operators in } \text{An}(k) \text{ with order } \leq i \right\}$.
We are going to exclusively consider $B$. 

\textit{Reason.} Each $B_i$ is finite dim. $k$-vector space.

\[ \text{Filtration} \Rightarrow \text{grade algebra}. \]

Suppose $R$ is a $k$-alg., $\mathcal{F}$ a filtration.

\[ \text{Gr}^\mathcal{F}(R) = \bigoplus_{i \geq 0} F_i/F_{i-1}. \quad (F_{-1} = 0) \]

Direct sum of $k$-vector space. How about multiplications? Let $F(i) = F_i/F_{i-1}$. Define it Naturally!

Not hard to see: it is well defined. Extend by $k$-linear.

What is $\text{Gr}^B(\text{An}(k))$, $B$ the Bernstein-filtration.

\textit{Theorem:} $\text{Gr}^B(\text{An}(k)) = \text{poly}$ over $k$ in $2n$ variables.

$* \ x_i d_i - d_i x_i = 1, \quad F_{\mathbb{R}}/F_{\mathbb{R}-1}$ send $-1$ to $0$, so $x_i d_i = d_i x_i$, comm!
It takes some work to show there's no other relations. Argue by contradiction.

* $A_n(k)$ is interesting because $[x_i, d_i] = 0$.

So the filtration is not that interesting ... Maybe I'm wrong ...

* Some "1" is killed, some not.

$x_i d_i - d_i x_i = -1$, is killed. The reason is, when you calculate "$x_i d_i - d_i x_i$", implicitly there is a degree "2" in the formula. Everything with a lower degree is killed. This gives a way of killing stuff. Safe without destroying the structure.

In some sense we modulate the "1" in a different way.

* What if the relation is, say $x_i^2 d_i - d_i x_i = 1$.

And we want to kill the "1"? Let $R_2 R_3$ shape the same grade?

Either, find a "grade", so $x_i^2$ and $d_i x_i$ have the same grade. Or, find away to kill "1" without using grades.