1 Assignment 3. Math 8401

1. Let $\sigma(x)$ denote the step function, defined as $\sigma(x) = 1$, $x \geq 0$ and $\sigma(x) = 0$, $x < 0$.

Consider the complete, orthonormal family $\{1, \cos nx, \sin nx\}_{n=1,2,...}$ (It is convenient to use the notation $\{\phi_i(x)\}$ to denote the sequence of functions).

- Write down the first four partial sums $s_1(x), s_2(x), s_3(x), s_4(x)$ of the Fourier series of $\sigma(x)$, $x \in [-\pi, \pi]$.
- Plot the partial sums.
- For each one of these approximations, calculate the $L^2$-error.
- Determine how many terms are needed to achieve a relative error $\frac{\|\sigma(x) - s_n(x)\|}{\|\sigma(x)\|} < 0.001$.

Hint: To calculate the absolute error, use Parseval's relation

$$\|\sigma - \sum_{i=1}^{n} \alpha_i \phi_i\|^2 = \|\sigma\|^2 - \sum_{i=1}^{n} |< \sigma, \phi_i>|^2.$$ 

Observe that the approximations overshoot $\sigma(x)$ in a neighborhood of $x = 0$, and the overshoot is not improved regardless of how many terms are taken in the approximation. This behavior of Fourier series near a discontinuity is called the Gibbs' phenomenon.

You may use Matlab or Mathematica for the calculations.

2. Consider the boundary value problem for Laplace’s equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad (x,y) \in \Omega,$$

$$u(x,0) = 0, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u(\pi,y) = y(\pi - y), \quad 0 < x < \pi, \quad 0 < y < \pi,$$

where $\Omega$ represents the square

$$\Omega = \{0 < x < \pi, \ 0 < y < \pi\}.$$

1. Suppose that $u(x,y)$ is a smooth solution of the boundary value problem. Find the maximum value of $u$ in $\bar{\Omega}$. [Hint: use the maximum principle].

2. Solve the boundary value problem by the method of separation of variables.

This assignment is due on Friday, November 1.

For the topic of Fourier series and separation of variables, please, look at the book by Peter Olver Introduction to Partial Differential Equations. It is available online, through the Math Library. You may also check the Library and materials online for other references.