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1 Mathematical Modeling: Introductory Remarks.

Applied mathematics deals with problems arising in the sciences, engineering and social sciences. Starting with a word problem, the goal is to give it a mathematical structure, mostly in terms of equations, analyze these equations, set them in a computational framework, and come up with quantitative results on the original problem. A validation process should be put in place to evaluate whether the results obtained accurately reflect the original problem.

The task of the applied mathematician may be summarized as follows:

- Consider problems emerging from science, engineering, medicine, social sciences, and, in general from real life.

- Give them a mathematical structure as appropriate, for instance, using the laws of physics (such as balance laws, mass, linear momentum, energy, ...), or make reasonable assumptions motivated by the experiments in question, or by whatever information is available on the problem. Once the model is built, it is very important to examine how it can be transported to problems that have emerged from very different situations.

- Apply methods of analysis to study the mathematical model at hand. These methods may relate to differential equations (ordinary, partial, stochastic,...), linear algebra, statistics,... In many occasions, new mathematics have emerged from the process of solving a real life problems. For instance, calculus emerged from the the study of gravity and planetary motion; the Maxwell equations and their analysis resulted from the study of electromagnetic phenomena and its applications.

- Cast the mathematical models in a computer amenable form. In problems formulated as systems of differential equations, this process typically involves discretization of space and time. Such discrete models are then analyzed by numerical methods, that are subsequently processes in a computer.

- Validation and revision of the computer generated data in terms of the original problem, for instance, comparing the results to experimental measurements.

1.1 Examples

The equation of the harmonic oscillator shown in figure 1 is

\[ m \frac{d^2x}{dt^2} + kx = 0. \]  

(1)

The general solution is

\[ x(t) = A \cos \sqrt{\frac{k}{m}} t + B \sin \sqrt{\frac{k}{m}} t, \]  

(2)

where \( A \) and \( B \) are constants that depend on the prescribed initial data.

The equation for the harmonic oscillator can be generalized to include friction \((c > 0\) denotes the friction coefficient), and also the presence of an external force \( F = F(t) \):

\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F(t). \]  

(3)
Heat equation. Let $D$ be a bounded domain in $\mathbb{R}^3$, with smooth boundary $\partial D$. The equation giving the distribution of temperature $u = u(x,t)$ in $D$ is

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T,$$

(4)

$\rho$ denotes the density of the material, $c$ the specific heat capacity, $k$ the conductivity. The independent variables are space $x \in D$, and time $t \geq 0$. The unknown function $T = T(x,t)$ denotes temperature.

To solve this equation, initial and boundary conditions need to be specified. The latter could be isothermal conditions, that is, the temperature is prescribed on the boundary, or flux conditions when the amount of heat going through the boundary is given.

Both, the linear oscillator equation and the heat equation are linear.

Topics on ordinary differential equations that we will study:

- Initial value problems for nonlinear, second order, and also special higher order equations. Analyze the evolution of the solution with time and its meaning. In the former case, we will study energy methods and the phase plane. One of the models that we will analyze is the nonlinear pendulum equation. We will also study some equations of third order, such as the Lorenz system, and population models such as the evolution of HIV.

- In many applications, the equations governing phenomena of interest contain one or more parameters. Consequently, solutions will also depend on such parameters. When there is a scale separation among parameters, perturbation methods are called for. We will study regular and singular perturbation methods. Here is a simple example of an ordinary differential equation that can be explicitly solved (rare!!):

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 0.$$  

(5)

We can see that the solution of this initial value problem is

$$y = \tan x.$$  

(6)
Can we use this information to solve the modified equations

\[
\frac{dy}{dx} = 1 + (1 + \epsilon)y^2, \quad |\epsilon| << 1.
\]  

(7)

Again, this latter problem has also an exact solution. How does the solution depend on the parameter \(\epsilon\)? How do we solve problems for which there is no exact solution? The answer is provided by perturbation methods.

- Perturbation methods and stability, such as the normal mode analysis, and eigenvalue problems.
- Boundary value problems and bifurcation.

The heat equation is a statement of balance of energy. Balance equations are very important in physics. We will present a derivation of the heat equation in terms of balance of energy.

The heat equation is also associated with diffusive processes (e.g., as when salt is dissolved in water). From this point of view, the equation is associated with stochastic phenomenon. We will also study the heat equation in such a context.

Prior to developing mathematical methods to solve certain problems, we will explore information that can be obtained on a problem from purely common sense.

2 Dimensional Analysis and Scaling Laws

Let us discuss the following example\(^1\) When we ride a bike, we notice that the force of air resistance is positively related to the speed and to the cross-sectional area (skinny versus broad rider). The force exerted upon a moving object opposing its motion is known as drag force.

The understanding of such a force is essential in the design of airplanes, cars, bicycles, boats, and any moving objects, aiming to the reduction of drag.

We want to find an equation that relates the force \(F\) with the velocity \(v\) and the area \(A\). We could write a prototype equation such as \(F = f(A, v)\). However, since the force involves mass, the equation cannot depend on \(v\) and \(A\) only. So, let us start with the equation

\[
F = f(\rho, A, v)
\]

where \(\rho\) denotes air density, and with \(f\) the relation to be determined. So, let us start with an equation of the form

\[
F = K \rho^x A^y v^z,
\]  

(8)

where \(K\) is a constant without units (i.e., a dimensionless constant). We use the symbols \(M\), \(L\) and \(T\) to denote mass, length and time, respectively.

Let us set up the dimensional equation associated with (8):

\[
MLT^{-2} = (ML^{-3})^x(L^2)^y(LT^{-1})^z.
\]  

(9)

\(^1\)Sam Howison
Equating the exponents of the three above quantities, we arrive at
\[ x = 1, \quad y = 1, \quad z = 2. \]
So, the force equation becomes
\[ F = k\rho Av^2. \]

**Remark.** Understanding scaling can help us to build small scale models of large phenomenon, such as wind tunnels to model airplanes.

We will revisit the previous problem at the end of the section. In particular, notice that we have omitted a very important ingredient in the derivation: the viscosity of the air.

### 2.1 The yield of a nuclear explosion by G.I. Taylor

G.I. Taylor (1940’s, Cambridge University) computed the energy yield of the first atomic explosion (New Mexico, 1945) after viewing the photographs of the spread of the fireball, \(^2\) \(^3\). He assume that there exists a physical law of the form
\[ g(t, r, \rho, E) = 0. \]
Here
- \( r \) denotes the radius of the front at time \( t \),
- \( \rho \) is the initial air density,
- \( E \) is the energy realised by the explosion.

We first ask how many dimensionless groups we can form with the quantities \( \{t, r, \rho, E\} \)? We find that
\[ \frac{r^5 \rho}{t^2 E} \]
is dimensionless and that there are no other independent dimensionless quantities that we can form with \( \{t, r, \rho, E\} \).

By the \textit{Pi-Theorem} (any physical law has a dimensionless form), we rewrite the original equation as
\[ f\left(\frac{r^5 \rho}{t^2 E}\right) = 0, \]
that is, \( f \) is a function of a single variable. Note that the solution corresponds to a root \( C \) (constant) of the previous equation. So,
\[ \frac{r^5 \rho}{t^2 E} = C, \]
which implies that
\[ r = \left(\frac{C \rho t^2}{E}\right)^{\frac{1}{5}}. \]

\(^2\) Sam Howison

This last relation is known as a scaling law and it states how the radius of the fireball grows with time.

\[ r \approx t^{\frac{2}{5}}. \]

It is confirmed by experiments and photographs.

### 2.2 The Nonlinear Pendulum
Sam Howison, page 32.

### 2.3 Scaling

Scaling is a procedure that reduces the original model, expressed in terms of dimensional variables, to one formulated in terms of the dimensionless variables. Scaling unveils the relative size of the parameters, reducing their number.

Let us consider the population model stating that the rate of growth of the number of individuals \( p \), at time \( t \), is proportional to the current population

\[
\frac{dp}{dt} = rp(t), \quad p(0) = p_0,
\]

where \( p_0 \) is the population at the beginning of the count; \( r > 0 \) is the growth rate, a quantity that has dimensions of \( t^{-1} \), where \( t > 0 \) denotes time. This model, known as the classical Malthus model (Thomas Malthus, who lived during the period 1766-1843, was an English essayist and one of the first individuals to study demographics and food supply).

This model predicts an exponential growth of the population, \( p(t) = p_0 e^{rt} \), which it is obviously not realistic at all. The model neglects the competition effects among individuals of a population group and also the limitations to growth. A modification of Malthus model, known as the logistics model, incorporates competition by accounting for the number of encounters, \( p^2 \), and also for the limitations of the environment to support unlimited growth:

\[
\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right), \quad p(0) = p_0.
\]

Here \( K > 0 \) denotes the carrying capacity, which is the maximum number of individuals that the ecosystem can sustain.

Note that the problem has three parameters: \( K, p_0, r \). Let us rewrite the model in terms of dimensionless variables, time and population. Let

\[ \tau = tr, \quad P = \frac{p}{K}. \]

(We could also scale \( p \) with \( p_0 \)). The equation becomes

\[
\frac{dP}{d\tau} = P(P - 1), \quad P(0) = \alpha,
\]

where \( \alpha = p_0 / K \). The problem can be solved by separation of variables, giving

\[ P(\tau) = \frac{\alpha}{\alpha + (1 - \alpha)e^{-\tau}}. \]
Note that
\[ \lim_{\tau \to \infty} P(\tau) = 1. \]

The population grows up to reaching the carrying capacity.

Note that the scaled equation only contains one parameter, \( \alpha \), down from the 3 constants of the original problem.

### 2.4 The Navier Stokes equations and the Reynolds number

The flow of an *incompressible viscous fluid* is governed by the *Navier-Stokes* equations for the velocity field \( \mathbf{v} \) and the pressure \( p \) of the fluid,

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \mu \Delta \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0,
\]

where \( \mu > 0 \) is the *viscosity* of the fluid, and \( \rho > 0 \) is the *density*; both such quantities are constant. The pressure \( p \) is the Lagrange multiplier associated with the incompressibility constraint.

Let us introduce the scaled (dimensionless) variables

\[
\tilde{t} = \frac{t}{T}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{v} = \frac{v}{U},
\]

where \( L, T, U = \frac{L}{T} \) and \( P \) denote typical *length*, *time*, *velocity* and *pressure* scales, respectively.

Obviously, out of the three quantities \( L, T, U \) only two are independent. Likewise, \( P \) cannot be chosen independently from other parameters of the problem. We will determine \( P \) in terms of other parameters of the model with the goal of obtaining the simplest possible coefficients. Recall that \( [P] = \frac{\text{force}}{\text{area}} \).

Now, let us write (12) in terms of the dimensionless variables (after dividing through by \( \rho \)) denoted with an upper bar) to obtain

\[
\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + \frac{UT \rho}{L \rho} (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} = -\tilde{\nabla} \tilde{p} + \frac{\mu T}{L^2 \rho} (\tilde{\Delta} \tilde{\mathbf{v}}).
\]

(13)

Let us choose \( P \) so that

\[
\frac{PT}{LU \rho} = 1.
\]

This gives

\[
\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} = -\tilde{\nabla} \tilde{p} + \frac{\mu T}{L^2 \rho} (\tilde{\Delta} \tilde{\mathbf{v}}).
\]

(14)

We define the *Reynolds* number as

\[
\text{Re} = \frac{L^2 \rho}{\mu T} = \frac{\rho UL}{\mu}.
\]

(15)

So, we finally write

\[
\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} = -\tilde{\nabla} \tilde{p} + \frac{1}{\text{Re}} (\tilde{\Delta} \tilde{\mathbf{v}}).
\]

(16)
Note that, although the original equation involved several parameters, the scaled nondimensional version depends only on one parameter, Re.

This allows us to simplify the equation at the limits of small and large Reynolds number. In particular, the case of Re large, corresponds to small viscosity $\mu$ (large compared with $UL$), in which case, we can replace (16) with

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p,$$  \hspace{1cm} (17)

$$\nabla \cdot v = 0.$$  \hspace{1cm} (18)

These are known as Euler equations of inviscid fluid. (We dropped the 'bar' notation.)

**What about the limit Re small?**

We get the Stokes problem

$$0 = -\nabla p + \frac{1}{Re}(\triangle v), \quad \nabla \cdot v = 0.$$  \hspace{1cm} (19)

### 2.5 The drag problem revisited

Let us return to the problem of determining the drag force on the bike rider. Pioneering studies of this problem go back to Stokes (19th century, Ireland and the UK), who studied the motion of an oscillating surface on water. Let us consider the simpler problem of a sphere moving in a fluid (air, water, ...). As in the case of the bicycle, we postulate that the drag force

$$F = f(R, v, \rho, \mu),$$  \hspace{1cm} (20)

where $R$ denotes the radius of the sphere, $v$ its velocity, and $\rho$ and $\mu$ the density and viscosity of the fluid, respectively. A dimensional analysis of the previous equation allows us to write

$$[F] = [R^a v^b \rho^c \mu^d],$$  \hspace{1cm} (21)

that is

$$MLT^{-2} = L^{a+b-3c-d}T^{-b-d}M^{c+d}.$$  \hspace{1cm} (22)

Equating both sides of the equation gives

$$a + b - 3c - d = 1,$$  \hspace{1cm} (23)

$$-b - d = -2,$$  \hspace{1cm} (24)

$$c + d = 1.$$  \hspace{1cm} (25)

We find that

$$a = 2 - d = b, \quad c = 1 - d.$$  \hspace{1cm}

Hence, a possible expression of $F$ is

$$F = \alpha \rho R^2 v^2 \left( \frac{\mu}{\rho v R} \right) = \alpha \rho R^2 v^2 (\text{Re})^{-d},$$  \hspace{1cm} (26)

---

4 M.H. Holmes, section 1.2
where $\alpha$ as well as $d$ are arbitrary constants. The fact that $\alpha$ and $d$ are arbitrary allows us to obtain more (general) solutions to the problem:

$$F = \rho R^2 v^2 \left[ \alpha_1 (\text{Re})^{-d_1} + \alpha_2 (\text{Re})^{-d_2} + \ldots \right].$$

(27)

In general, we can state

$$F = \rho R^2 v^2 G(\text{Re}^{-1}).$$

(28)

The function $G$ has been experimentally obtained for different fluids.

3 Models Derived from Balance Laws

We mentioned that some mathematical models, especially those coming from mechanics, can be formulated in terms of balance laws. The next example presents a statement of balance of energy leading to the heat equation.

3.1 Equation of Balance of Energy

We derive an equation governing the flow of heat in a homogeneous, isotropic and continuous solid.

This picture represents a bounded domain $\mathcal{D} \subset \mathbb{R}^3$, with smooth boundary, $\partial \mathcal{D}$. The vector $\mathbf{n}$ denotes the unit outward normal to the boundary, and $\mathbf{q}$ represents the heat flux vector. In addition to $\mathbf{q}$, we introduce the energy density $E(\mathbf{x}, t)$ (energy per unit volume at a point $\mathbf{x}$ and at time $t$). This energy is associated with random molecular motion. Recall that $\mathbf{q} \cdot \mathbf{n}$ represents the amount of energy (heat) going out of the domain across the boundary per unit area and per unit time. (So, $-\mathbf{q} \cdot \mathbf{n}$ is the influx).

The following equation is the statement of balance of energy in the body $\mathcal{D}$:

$$\frac{d}{dt} \int_{\mathcal{D}} E(\mathbf{x}, t) \, d\mathbf{x} = -\int_{\partial \mathcal{D}} \mathbf{q} \cdot \mathbf{n} \, ds.$$  

(29)

Applying the divergence theorem to the surface integral gives

$$\frac{d}{dt} \int_{\mathcal{D}} E(\mathbf{x}, t) \, d\mathbf{x} + \int_{\mathcal{D}} \nabla \cdot \mathbf{q} \, d\mathbf{x} = 0$$

(30)
Note that this statement of balance of energy can be applied to any part of the body $\mathcal{D}$. It, then, follows that the integrand is identically zero. (Here we assume that the integrand is continuous, in which case, the localization theorem applies). Hence,

$$\frac{\partial}{\partial t}E(x, t) + \nabla \cdot q = 0.$$ \quad (31)

We observe that this equation has more unknowns than variables. So, we need to specify constitutive equations, that is, relations between $E$ and $q$ so as to get a single unknown field. Constitutive equations also specify the type of material under consideration. In this case, we assume that

$$E(x, t) = \rho c T(x, t),$$ \quad (32)

$$q(x, t) = -k \nabla T(x, t).$$ \quad (33)

The first equation gives the energy of the body as function of the absolute temperature. This is consistent with temperature as measure of random molecular motion. The second equation is Fourier Law of heat conduction expressing the fact that heat flows from hot to cold. Here,

- $\rho > 0$ denotes the material mass density, and $c > 0$ the specific heat capacity (the amount of heat required to raise the temperature of unit of mass of the material, at temperature $T$, by one degree),

- $k > 0$ represents the heat conductivity.

So, substituting the previous constitutive relations into the equation of balance of energy (local form), we get the heat equation:

$$\frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T), \quad \kappa = \frac{K}{\rho c},$$ \quad (34)

where $\kappa$ is the thermal diffusivity of the material. Examples of thermal conductivity values in $m^2/sec$ units (square meters per second):

- water: $1.4 \times 10^{-7}$,
- air : $2.2 \times 10^{-5}$,
- gold $1.27 \times 10^{-4}$ (best heat conductor).

### 3.2 A priori estimates on the heat equation: Uniqueness of solution

After having derived the heat equation, we give an example on how to use properties of the equation to obtain information on solutions. Let us consider the following initial boundary value problem. $\mathcal{D} \subset \mathbb{R}^n$, $n = 1, 2$ is an open, bounded set with smooth boundary $\partial \mathcal{D}$, and $\partial \mathcal{D}_i \subset \partial \mathcal{D}$, satisfying $\partial \mathcal{D}_1 \cup \partial \mathcal{D}_2 = \partial \mathcal{D}$, $\partial \mathcal{D}_1 \cap \partial \mathcal{D}_2 = \emptyset$. are as

$$\frac{\partial u}{\partial t} - \kappa \triangle u = 0, \quad \text{in } \mathcal{D}$$ \quad (35)

$$u = 0, \quad \text{on } \partial \mathcal{D}_1,$$ \quad (36)

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \quad \text{on } \partial \mathcal{D}_2.$$ \quad (37)

$n$ denotes the outward unit normal vector to the boundary.
Theorem 3.1. Suppose that $u : D \longrightarrow \mathbb{R}$ is a solution of the initial boundary value problem (35)-(37). Then it satisfies the following energy relation:

$$\frac{1}{2} \frac{d}{dt} \int_D u^2 \, dx = -\kappa \int_D |\nabla u|^2 \, dx.$$  \hspace{2cm} (38)

Consequently,

$$\int_D u^2(x, t) \, dx \leq \int_D u^2(x, 0) \, dx, \quad t \geq 0$$  \hspace{2cm} (39)

holds.

Proof. We first multiply (35) by $u(x, t)$ and integrate over $D$. Next, we apply the following vector identity to the term containing $\triangle u$:

$$u \triangle u = \nabla \cdot (u \nabla u) - |\nabla u|^2.$$  \hspace{2cm} (40)

These yield the equation

$$\frac{1}{2} \frac{d}{dt} \int_D u^2 \, dx + \kappa \int_D |\nabla u|^2 \, dx - \int_{\partial D} u \frac{\partial u}{\partial n} \, dS = 0.$$  \hspace{2cm} (41)

Note that we have applied the divergence theorem to the volume integral of the term $\nabla \cdot (u \nabla u)$, resulting in the surface integral in (41). The result follows by applying the boundary conditions (36) and (37) on this surface integral term. \qed

From this energy identity, uniqueness of solution of the inhomogeneous, linear heat equation follows. Specifically, consider the problem

$$\frac{\partial u}{\partial t} - \kappa \triangle u = f, \quad \text{in} \ D$$  \hspace{2cm} (42)

$$u = \bar{u}, \quad \text{on} \ \partial D_1,$$  \hspace{2cm} (43)

$$\frac{\partial u}{\partial n} = g, \quad \text{on} \ \partial D_2,$$  \hspace{2cm} (44)

where $f$, $\bar{u}$ and $g$ are prescribed, smooth fields.

Corollary. Let $u$ be a solution of the initial boundary value problem (42)-(44). Then it is unique. The proof follows by contradiction, that is, assuming that there are two distinct solutions $u_1$ and $u_2$ to the the problem (42)-(44).

Note. A priori estimates of solutions of the heat equation allow us to obtain properties of solutions even when we do not have expressions for it, that is, before even finding the solution (if it exists). Another tool to determine solution properties, a priori, is the maximum principle. Its proof relays entirely on calculus.

Theorem 3.2. Let $\Omega \subset \mathbb{R}^3$ be an open, bounded domain with smooth boundary $\partial \Omega$. Suppose that $u(x, t)$ is a smooth solution of the heat equation

$$u_t(x, t) = \triangle u(x, t) + f(x, t), \quad x \in \Omega, \ t > 0,$$  \hspace{2cm} (45)

where $f(x, t)$ is a smooth function, such that $f(x, t) \leq 0$ in $\Omega \times [0, T]$, for some $T > 0$. Then the maximum of $u(x, t)$ is achieved either on the domain boundary $\partial \Omega$ or at time $t = 0$. 

10
Proof. We first prove the theorem for \( f < 0 \). First of all, since \((\Omega \cup \partial \Omega) \times [0, T]\) is a bounded, closed set and \( u(x, t) \) is continuous, \( u \) must achieve its maximum and its minimum in \((\Omega \cup \partial \Omega) \times [0, T]\). However, the extreme values could occur either in the interior \((\Omega \times (0, T))\) or on the boundary \((\partial \Omega \times \{0, T\})\).

We proceed by contradiction and assume that \( \max u(x, t) := u(x^*, t^*) \), where \((x^*, t^*)\) is an interior point of \( \Omega \times [0, T] \). Thus,

\[
u_t(x^*, t^*) = 0, \quad \text{and} \quad \triangle u(x^*, t^*) \leq 0,
\]

which is not allowed because \( f(x^*, t^*) < 0 \).

Next, we assume that \( u(x, T) \) is the maximum, that is, the maximum occurs at \( t = T \) and \( x \in \Omega \): that is, \( u_t(x, T) \leq 0 \) and \( \triangle u(x, T) \geq 0 \), which, again, gives a contradiction. Hence, the conclusion of the theorem follows for \( f < 0 \).

Let us now assume that \( f(x, t) \leq 0 \) in \( \Omega \times (0, T) \). Since \( \Omega \) is bounded, it can be enclosed in a sphere of radius \( R > 0 \), for \( R > 0 \) sufficiently large. (Without loss of generality, we assume that the sphere is centered at the origin of the coordinate system \((x, y, z)\).) Let \( \epsilon > 0 \) and \((x, y, z)\) be a point in the sphere. Define

\[
v(x, t) = u(x, t) + \epsilon(x^2 + y^2 + z^2) \leq u(x, t) + \epsilon R^2
\]

Note that

\[
u_t = v_t, \quad \text{and} \quad \triangle v = \triangle u + 6\epsilon,
\]

so that \( v \) satisfies the heat equation

\[
v_t = \triangle v - 6\epsilon + f(x, t).
\]

Since \( f(x, t) - 6\epsilon < 0 \), by the first part of the theorem we infer that \( v \) satisfies the maximum principle. That is,

\[
\max_{\Omega \times [0, T]} v(x, t) = \max_{\partial \Omega \cup \{t = 0\}} v(x, t).
\]

Equivalently

\[
\max_{\Omega \times [0, T]} u(x, t) + \epsilon(x^2 + y^2 + z^2) = \max_{\partial \Omega \cup \{t = 0\}} u(x, t) + \epsilon(x^2 + y^2 + z^2).
\]

Taking the limit as \( \epsilon \to 0 \) in the previous expression and using the fact that \( x^2 + y^2 + z^2 \leq R^2 \), we arrive at the statement

\[
\max_{\Omega \times [0, T]} u(x, t) = \max_{\partial \Omega \cup \{t = 0\}} u(x, t).
\]

Remark. Note that the maximum principle establishes that, in the presence of a heat sink in the domain \((f \leq 0)\), the temperature in the interior of \( \Omega \) cannot be higher than the boundary temperature or the initial one. Consequently, interpreting \( u \) as the absolute temperature, the maximum principle asserts the positivity of the temperature in the domain, for all time.
Let us conclude this subsection with an example that illustrates the type of information that we can get from energy estimates.

**Example.** Consider the partial differential equation

\[
\frac{\partial u}{\partial t} = \kappa \Delta u + au \quad \text{for} \quad x \in \Omega \subset \mathbb{R}^3,
\]

\[
u(x, 0) = u_0(x) \quad \text{in} \quad \Omega,
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \(a\) and \(\kappa > 0\) are constant, and \(\Omega\) is open and bounded.

1. Write down the energy law of the problem.

2. Find the long-time behavior of the solutions, that is, the limit \(u(x, t)\) as \(t \to \infty\).

We proceed as in theorem (3.1) to arrive at the inequality

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx = -\kappa \int_{\Omega} |\nabla u|^2 \, dx + a \int_{\Omega} u^2 \, dx. \tag{47}
\]

If, in addition, \(a \leq 0\), it follows that

\[
\int_{\Omega} u^2(x, t) \, dx \leq \int_{\Omega} u^2(x, 0) \, dx, \quad t \geq 0. \tag{48}
\]

In order to get a sharper result, we appeal to the following change of variable

\[
v(x, t) = e^{-at} u(x, t).
\]

A simple calculation shows that \(v_t = (-au + u_t) e^{-at}, \Delta v = e^{-at} \Delta u\) and \(v(x, 0) = u(x, 0)\). Substituting these expressions into equation (47), we find that \(v\) satisfies the equation

\[
v_t(x, t) = \kappa \Delta v(x, t).
\]

Hence

\[
\int_{\Omega} v^2(x, t) \, dx \leq \int_{\Omega} v^2(x, 0) \, dx, \quad t \geq 0,
\]

which, in terms of the original function \(u\) becomes

\[
e^{-2at} \int_{\Omega} u^2(x, t) \, dx \leq \int_{\Omega} u^2(x, 0) \, dx, \quad t \geq 0.
\]

Hence

\[
\int_{\Omega} u^2(x, t) \, dx \leq e^{2at} \int_{\Omega} u^2(x, 0) \, dx, \quad t \geq 0.
\]

Taking limits as \(t \to \infty\) on both sides of the inequality, we get

\[
\lim_{t \to \infty} \int_{\Omega} u^2(x, t) \, dx = 0, \quad t \geq 0.
\]

By continuity of \(u\) and the localization theorem, we conclude that \(\lim_{t \to \infty} u(x, t) = 0, \forall x \in \Omega\).
4 Phase plane analysis: the nonlinear pendulum

The phase plane method applies to autonomous second order ordinary differential equations and also to first order systems of two autonomous equations. The phase plane method gives the qualitative behaviour of all solutions. Here, we present the method as it applies to the nonlinear pendulum equation. We first consider the motion of a pendulum without friction forces. In its dimensionless form is

\[
\frac{d^2\theta}{dt^2} + \sin \theta = 0, \quad (49)
\]

\[
\theta(0) = a, \quad \frac{d\theta}{dt}(0) = b. \quad (50)
\]

Here \( \theta \) denotes the angle that the pendulum arm makes with the vertical, and \( a, b \) are constants.

4.1 Energy

Let us multiply equation (49) by \( \frac{d\theta}{dt} \) and integrate term by term the resulting expression:

\[
\frac{d\theta}{dt} \left( \frac{d^2 \theta}{dt^2} + \sin \theta \right) = 0,
\]

\[
\frac{1}{2} \dot{\theta}^2 - \cos \theta = E, \quad (51)
\]

where \( E \) is an arbitrary constant. (Notation: \( \dot{\theta} = \frac{d\theta}{dt} \)). In terms of the boundary conditions \( (a, b) \), we have that \( E = \frac{1}{2}b^2 - \cos a \). We interpret equation (51) as the statement of conservation of energy. Indeed, the first term of the left hand side is the kinetic energy \( KE := \frac{1}{2} \dot{\theta}^2 \), and the second term is the potential energy, \( PE := U(\theta) = -\cos \theta \), so that equation (51) states that \( KE + PE = \) constant. So,

\[
\frac{1}{2} \omega^2 + U(\theta) = E, \quad \omega := \frac{d\theta}{dt}. \quad (52)
\]

Curves \( \{\theta(t), \omega(t)\} \) satisfying (52) are called the orbits or trajectories of the system with total energy \( E \). Solutions \( (\theta(t), \dot{\theta}(t)) \) of the initial value problem, with initial conditions satisfying \( E = \frac{1}{2}b^2 - \cos a \), if they exist, are points of the trajectory corresponding to \( E \). The phase plane \( (\omega, \theta) \) consists of all the orbit plots. In the phase plane, locally \( \omega = \omega(\theta) \), according to the differential equation obtained from (52),

\[
\omega \frac{d\omega}{d\theta} + \sin \theta = 0. \quad (53)
\]

Correction: In the adjoint \( U(\theta) \)-graph, the entries in the vertical axis should be divided by 2: \( 0, \pm 0.5, \pm 1, \pm 1.5, \ldots \). Accordingly, \( E_3 = 1, E_0 = -1 \).

Problem 1. Show that the orbits corresponding to \( E = E_1 \) in the graph of \( U(\theta) \) are closed. So, the corresponding solutions of the initial value problem (49), if they exist, are periodic. If, in addition, the initial data \( \theta(0) = a \) belongs to the potential well around \( \theta = 0 \), then the solutions represent oscillations of the pendulum about the vertical position.
Problem 2. Describe the qualitative behavior of solutions with energy $E = E_4$.

Problem 3. What is the behavior of solutions corresponding to $E = E_0$?

Symmetry properties of the energy. We point out that equation (52) is invariant under the transformations

$$
\theta \rightarrow -\theta, \quad \omega \rightarrow -\omega. 
$$

Consequently, the study of the phase plane $(\theta, \omega)$ reduces to the positive quadrant. Moreover, due to the $2\pi$-periodicity of the potential energy, it is sufficient to restrict the phase plane domain to \( \{0 \leq \theta < 2\pi, \omega \geq 0\} \).

4.2 Phase Plane

We now use the differential equation (53) to determine properties of the orbit in the phase plane. The following properties are easy to verify. Fix $\omega > 0$. Then

1. $\omega' = 0$ at $\theta = 0, \pi$. This implies that the orbits have a horizontal tangent on the $\omega-$axis.

2. $\omega' < 0$ for $0 < \theta < \pi$, and so $\omega$ decreases in that interval.

3. $\omega' > 0$ for $\pi < \theta < 2\pi$, and, so $\omega$ increases in that interval.

4. $\lim_{\omega \to 0} |\omega'| = \infty$, for $\theta \neq 0, \pi$. Hence the orbits have a vertical tangent in the $\theta-$axis.

Local study of the equilibrium points. Let us now consider the behavior of the orbits near $(0, 0)$ and $(\pi, 0)$.

Linearization of equation (53) about $\theta = 0$ and its subsequent integration give

$$
\omega \omega' + \theta = 0, \quad \omega^2 + \theta^2 = \text{constant}. 
$$
So, orbits near \((0,0)\) are circles. Equilibrium points with this property are called *centers*.

Likewise, we linearize about \((\pi,0)\) and integrate to get

\[\omega \omega' - \bar{\theta} = 0, \quad \omega^2 - \bar{\theta}^2 = \text{constant}, \quad \bar{\theta} := \theta - \pi.\]  

(56)

Hence, the orbits near \((\pi,0)\) are hyperbolas. In particular, choosing the arbitrary constant to be zero, the latter reduce to lines with slope \(\pm \frac{\pi}{4}\) through \((\pi,0)\). Equilibrium points as \((\pi,0)\) are labelled *saddle points*. They are unstable.

### 4.3 Separatrix orbits

Now, let us consider the entire phase plane \((\theta,\omega)\). Let us consider the closed orbits that contain the equilibrium point \((\pm \pi,0)\). It is easy to check that they also contain the point \((0,\pm 2)\), with energy \(E = 1\) in (52).

With the help of the energy equation (52), show the following properties of the orbits:

1. The separatrix orbits are closed.

2. Orbits of solutions with initial data \((a,b)\) and such that \(0 < a < \pi\) and \(|b| < 2\) satisfy \(|\theta| < \pi\) for all \(t \geq 0\). This indicates that the pendulum oscillates about \(\theta = 0\) for all time, and it never over turns.

3. Let \((a,b)\) as above, but with \(|b| > 2\). The trajectory will keep passing through the angles \(\theta = \pm \pi\), that is, will reach an instantaneous vertical rest position and will keep cycling around its support.

4. Note that the separatrix orbits separate the regions of the phase plane with behaviors as describe in items (2) and (3) above.

5. Why is \(b = 2\) so special? Give an answer in terms of the energy. Orbits corresponding to solutions with initial data inside the separatrix show behavior (2), while outside they behave as in (3).

### 5 Critical points of ODE systems. Stability analysis

Some theory and examples are available in the context of Matlab coming next.

### 6 Asymptotics: Regular Perturbations

We start with covering Chapter 2 of the book by Logan, *Applied Mathematics*. We will also examine several case studies.

**A swimming protozoan.** Consider a protozoan swimming in a viscous fluid. The motion of the flagellum is given by the equation

\[y = \epsilon \sin(kx - \omega t),\]
where $0 < \epsilon \ll 1$, and $k$ and $\omega$ are also constant. Find the fluid motion induced by the motion of the flagellum.

**Solution.** First of all, let us discuss the assumptions on the model.

- The coordinates $(x, y)$ are taken with respect to a system with origin in the protozoan, that is the observer of this motion protozoan/fluid moves with the protozoan, and so, it sees the fluid moving by.

- Think of the $x$ coordinate as a location in a fixed axis of the flagellum, and $y$ giving the displacement of the tip with respect to the axis.

- We assume that the motion is uniform with respect to the $z$-coordinate. That is, rather than a slim moving object, in fact, we have a corrugated board moving in the fluid.

- The fluid moves according to the Stoke’s equations.

According to Stoke’s law, the equations of motion of a viscous flow are given by

$$\mu \nabla^2 \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u}$ denotes the vector field of the fluid, and $p$ its pressure. The constant $\mu > 0$ represents the viscosity of the fluid.

**Remark.** Recall that in the scaling of the Navier-Stokes equations $Re = \frac{UL}{\mu} \ (U$: typical speed, $L$: typical length scale, $\rho$: viscosity) is the dimensionless parameter known as the Reynolds number. For small $Re$, the dimensionless form of the equations is

$$Re \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \nabla^2 \mathbf{u}.$$  

(58)

So, the Stokes problem comes at the limit $Re = 0$ (e.g., highly viscous fluid).

(Problem to be continued.) $\square$

### 7 Matlab

A this point, you need to set up your computational framework, reviewing the software of your preference. In class, we will mostly use Matlab. It is available at the CSE computational center and you can access it online. The first step is to work through a tutorial, if you have not done so before. Here are some tips that you may use, together with examples that we will work out. An excellent reference is *Matlab Guide* by Higham and Higham (3th edition, 2016). It is available at the Library.
7.1 Accessing Matlab: Running Demos and Tutorials

To access Matlab, you should type `matlab` at the prompt of your xterm window. (Please, be aware that Matlab is case sensitive, and to start running it, you need to type the word in low-case characters.) Matlab will then appear in your screen with the prompt `>>`. You may type `demo` to get an introduction to the program. It is helpful to explore `helpwin` and `helpdesk` as well. There are many Matlab tutorials available online. You may access some through the website [www.mathworks.com](http://www.mathworks.com), or carry out an Internet search for Matlab tutorials. Please, practice as much as possible with demos and tutorials.

2 Files and Directories

First of all, make sure that you are working in the right directory. It should be the one where you have stored the `matlab` (.m) and data files (when available).

To start Matlab type `matlab` and press `return`. The program will begin with the prompt `>>`. All Matlab commands will be given at this prompt.

If you have any questions about any Matlab command, you may use the on-line help. Typing `help` the prompt will give a list of all possible topics (there is also the command `help help`). Typing `help topic` gives information on `topic`. To get help on a specific function type `help function`.

As tests, type `help plot`, `help solve`, `help tan`, `help atan` and `help exp` and see the results.

3 Setting up variables

3.1 Scalar Variables

Suppose we want to set the value of the variable $x$ to be 9. Then, we type $x = 9$ at the Matlab prompt. Matlab returns

$$x = 9$$

indicating that now $x$ equals 9. To prevent Matlab from printing the value to the screen, we append a semicolon `;` to the command (i.e. $x = 9$).

1. Arithmetic operations such as $+,-,\times,\div$ are straightforward. As a test, let $x = 42$, $y = 24$ and calculate $x + y, x - y, x \times y, x/y$.

2. Powers are performed as in $x^2, x^{11}, \ldots$. Test $x^y$ from the previous example. Note that only 5 digits of the mantissa are shown. Matlab calculates with many more digits, though: type the command `format long` and then enter $x^y$ again. To go back to the shorter mantissa, type `format short`. Note that you can store $x^y$ in a variable $z$ by typing $z = x^y$. 

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3. Trigonometric functions have the expected terminology, \( \sin, \cos, \tan, \cot, \sec, \csc \), with \( \sin(x), \cos(x), \ldots \) giving the numerical values for a real number \( x \). The exponential function responds to the command \( \exp \). The natural logarithm function corresponds to \( \log \), and \( \log_{10} \) is the decimal logarithm. With the data provided above, compute \( \sin(y), \cos(y), \tan(y), \ldots \) with \( y \) given above. Typing the command \( \text{elfun} \) you will get a list of the elementary functions available in Matlab.

3.2 Vector Variables

One of the outstanding features of Matlab is the handling of vector variables. To enter a row vector \( x = (1, 2, 3) \) one types \( x=[1 \ 2 \ 3] \) at the Matlab prompt. The corresponding column vector is represented as \( y=[1 \ 2 \ 3]' \) or \( y=[1; \ 2; \ 3] \). As a test, type \( \text{size}(x), \text{size}(y), \text{length}(x), \text{length}(y) \) at the Matlab prompt and see the results. What are the roles of the functions \( \text{size} \) and \( \text{length} \)? (You may want to check \( \text{help size} \) and \( \text{help length} \).

In addition to providing matrix representation for linear algebra, one important role of the vector functions in Matlab is to replace \( \text{loop} \) statements in programs. For this, we need to introduce the \( \text{colon :} \) command. Typing \( k=1:n \) returns all positive integers between 1 and \( n \). Try entering \( n=10 \) at the Matlab prompt, press return, and then type \( k=1:n \). (Also, check \( \text{help colon} \).) Now, suppose that we want to calculate \( \sin(k\pi) \), for \( k = 1, \ldots, n \). We first type \( k=1:n \), press return, and then enter \( x=\sin(k*\pi) \). Test for \( n = 10 \).

A related problem is the calculation of \( \sin(x_k \pi) \), where \( x_k \) are the points \( 0 = x_1 < x_2 < \ldots < x_n = 1 \) of the uniformly spaced grid for the interval \( [0, 1] \) (with subinterval length \( 1/n \)). This is solved through the command \( \text{linspace} \). Typing \( z=\text{linspace}(a, b, n) \) creates a row vector \( z \) with

\[
z(k) = a + (k-1) \times (b-a)/(n-1), \quad \text{for} \quad k = 1, \ldots, n.
\]

Test by typing \( z=\text{linspace}(0,1,11) \). Next enter \( y=\sin(z*\pi) \), and compare the new vector \( y \) with the former \( x \).

By the way, notice that we keep using \( x, y, z \) to represent variables. To avoid confusion, once a variable is of no further use, we may remove it with the command \( \text{clear} \), by typing \( \text{clear} \ x \). If you type only \( \text{clear} \), all the variables will be removed. It is safer to remove one at a time.

4 Graphs

The most important command to generate graphs is the \( \text{plot} \) command. It can be used to generate two as well as three dimensional objects. The entries of the plot command are vectors:

\( \text{plot}(x, y) \) plots the vector \( x \) against the vector \( y \).

Example. The plot function can be used to \textit{join-the-dots} \( x - y \) plots. Typing:
\[ x = [1.22.12.34.25.56.78.9]; \\
y = [3.12.65.37.63.94.41.7]; \\
plot(x, y) \\
\]

produces a picture where the points \((x(i), y(i))\) are joined in sequence.

**Example.** Create the plot of \(y = \sin(x)\) over the interval \([0, \pi]\). Let us take the \(x\)-step size equals to \(.1\).

Type:
\[
x = 0: .1: \pi; \\
y = \sin(x); \\
plot(x, y) \\
\]

The graphics command **fplot** is a useful alternative to the **plot** command for functions. As a test, try typing
\[
fplot(\text{`tanh'}, [\text{-2}, \text{2}]) \\
\]
The command plot has many attributes. For instance you may want to print labels for the axes, plot multiple graphs in one set of axes, vary the thickness and properties of the graph.

Try repeating the two previous examples with the following plot commands:
\[
plot(x, y, \text{`LineWidth'}, 2) \\
plot(x, y, \text{`m--'}, \text{`LineWidth, 3, `MarkerSize'}, 5) \\
\]
To find out how to use all such attributes, type **help plot**.

### 4.1 Three–Dimensional Plots

The command **plot3** is the three-dimensional analogue of **plot**. Work out the following

**Example.** Draw the joint-the-dots curve by taking the points \((x(i), y(i), z(i))\) in order, where \(x(i), y(i)\) and \(z(i)\) are generated as follows:

\[
\begin{align*}
  t &= -4 : .005 : 4; \\
  x &= (1 + t^2) \sin(20t); \\
  y &= (1 + t^2) \cos(20t); \\
  z &= t; \\
\end{align*}
\]

\[
\begin{align*}
  \text{plot}(x, y, z) \\
  \text{grid on} \\
  \text{xlabel(\text{`x(t)'}, $\text{ylabel(\text{`y(t)'}, $\text{zlabel(\text{`z(t)'})} \\
  \text{title(\text{`plot3 example'}, $\text{FontSize, 14})} \\
\end{align*}
\]

The function **meshgrid** is very important in setting up data for three dimensional graphics. Given two vectors, \(x\) and \(y\), typing
\[ [X, Y] = \text{meshgrid}(x,y) \] yields two matrices, \( X \) and \( Y \), such that each row of \( X \) is a copy of \( x \) and each column of \( Y \) is a copy of \( y \).

Type the commands \textbf{help contour} and \textbf{help mesh} to learn about producing contour plots and wire-frame surface plots.

## 5 Programming in Matlab

### Relational operators

1. \( == \) Equal to
2. \( = \) not equal to
3. \( < \) Less than
4. \( > \) Greater than
5. \( \leq \) Less or equal to

### Logical operators

1. \( \neg \) Not (Complement)
2. \( \& \) And (True if both are true)
3. \( \mid \) Or (True if either or both operands are true)

The \textbf{for} and \textbf{while} statements in Matlab operate in a similar manner in other languages. They have the following form:

1. \textbf{for} (loop-variable = loop-expression) \hspace{2cm} \text{executable-statements}

2. \textbf{end}

3. \textbf{if} (logical expression) \hspace{2cm} \text{executable-statements}

4. \textbf{else} (logical-expression) \hspace{2cm} \text{executable-statements}

5. \textbf{end}

6. \textbf{while} (while-expression) \hspace{2cm} \text{ executable-statements}

7. \textbf{end}
The following example uses nested loops to generate a matrix.

**Example**

for \( i = 1 : 5 \)

\[
A(i, 1) = 1; A(1, i) = 1;
\]

end

for \( i = 2 : 5 \)

for \( j = 2 : 5 \)

\[
A(i, j) = A(i, j - 1) + A(i - 1, j);
\]

end

end

\( A \)

The **break** command is used to exit from a loop.

**Example**

for \( k = 1 : 100 \)

\[
x = \sqrt{k};
\]

if \( ((k > 10) \& (x - \text{floor}(x) == 0)) \)

break

end

end

**disp** command can be used to display text or a matrix.

**Example**

\( n=10; k=0; \)

while \( k <= n \)

\[
x = k/3;
\]

\[
disp([x \quad x^2 \quad x^3])
\]

\( k = k + 1; \)

end

6 Floating-point arithmetic and round-off errors

We express nonzero floating-point numbers in the form

\[
x = \pm (1 + f)2^e,
\]

with

\[
0 \leq f < 1 \quad \quad -1022 \leq e \leq 1023, \quad \text{and}
\]
The fraction $f$ is representable in the binary system using at most 52 bits. So,

$$0 \leq 2^{52} f < 2^{52}.$$  

Note that $2^{52} f$ is an integer. Double-precision floating-point numbers are stored in a 64-bit word, with 52 bits for $f$, 11 bits for $e$, and 1 bit for the sign of the number. The sign of $e$ is accommodated by storing $e + 1023$, which is between 0 and $2^{11} - 1$. Let us denote the minimum and maximum values of the exponent by $e_{\text{min}}, e_{\text{max}}$, respectively.

Note that in each binary interval $2^e \leq x \leq 2^{e+1}$, the numbers are equally spaced with an increment of $2^{e-52}$.

**Exercise.** Consider a binary system with $t = 3$ bits to represent the mantissa $f$, and with $e_{\text{min}} = -4$ and $e_{\text{max}} = 3$. Find the spacing between two consecutive numbers in the binary intervals $[1, 2]$ and $[\frac{1}{2}, 1]$. Repeat the exercise with $t = 5$.

**Notation.** Note the difference in notation of the exponent $e$ of the binary representation of $x$, with respect to the one used in class, where we write $e = e + 1023$.

One important quantity characterizing the accuracy of a computer is known as $\text{eps}$ ("machine epsilon"). It is the distance from 1 to the next floating-point number. For the previous model $\text{eps} = 2^{-t}$; for the 64-digit pc, $\text{eps} = 2^{-52} \approx 2.2204 \cdot 10^{-16}$. We can say that the round-off level is 16 decimal digits. Note that the maximum relative error incurred when the result of an arithmetic operation is rounded to the nearest floating-point number is $\frac{\text{eps}}{2}$.

The following exercise illustrates the effect off round-off error in adding and subtracting numbers.

**Exercise.** Plot the seventh degree polynomial:

$$x = .988 : .0001 : 1.012;$$

$$y = x^7 - 7x^6 + 21 \times x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1.$$  

Next, the polynomial $y = (x - 1)^7$ on the same interval. Discuss the plots that you obtain.

In plotting the first polynomial, observe that the $y$-axis scale factor is $10^{-14}$. The small values of $y$ are being computed by taking sums and differences of numbers as large as $35 \times 1.012^4$. There is severe subtractive cancellation.

## 7 Stability of Algorithms

A stable algorithm is such that, small changes in the initial input produce small changes in the result.

The following is an example of an unstable algorithm.

**Example.** Consider the recurrence algorithm

$$y_{n+1} = -4 \times h \times y_n + y_{n-1} + 2 \times h, \ y_0 = 1,$$

where $h > 0 \ y_1$ are specified. (Note that $y_1$ needs to be prescribed as well for the algorithm to be complete). This is a recurrence formula to approximate the solution of the differential equation

$$y' = -2y + 1, \ y(0) = 1.$$
This initial value problem has solution

\[ y(x) = 0.5(e^{-2x} + 1). \]

Given the integer \( N > 0 \), and setting \( h = \frac{1}{N} \), plot the approximate solution to the problem in the following cases:

\[ y_1 = 0.5(e^{-2h} + 1), \quad y_1 = 0.6, \quad y(1) = 0.9 \quad \text{and} \quad y(1) = 0.2 \]

First of all, we want to write Matlab code to implement the algorithm.

```matlab
%input N; y_1
%variables a, b, c, data; %data is a vector variable to store the output;

h=1/N; a=1; b=y_1;
data=zeros(1, N-2); n=1;
while n < N - 1;
c=-4*h*b+a+2*h; a=b; b=c; data(n)=data(n)+c; n=n+1; end
vector=[1 y_1 data];
x=1:N;
plot(x, vector, 'o', 'MarkerSize', 8)
hold on
plot(x, vector, 'LineWidth', 2)
hold off

The following commands calculate the exact solution, plot it, calculate the absolute error and draft its plot.

```
```matlab
exact=zeros(1, N);
p=0.5*(exp(-2*h)+1); k=1;
while k < N + 1
p=0.5*(exp(-2*k*h)+1);
exact(k)=exact(k) + p;
k=k+1;
end
hold on
plot(x,exact)
hold off
```

8 Integration

We can solve definite integrals such as

\[ \int_{a}^{b} g(x) \, dx \]
using Matlab. Although there are several approaches to the problem, the most efficient is with the command

\texttt{quad(function, a, b, tolerance)}. Let us look at the following example. To calculate

\[ \int_0^1 e^{-x^2} \, dx, \]

we first define the function by creating an M-file (\texttt{integrand.m}):\n
\begin{verbatim}
function y = integrand(x)
y = exp(-x^2);
\end{verbatim}

The integral calculated with .00001 tolerance is:

\begin{verbatim}
format long
integral = quad('integrand', 0, 1, .00001)
\end{verbatim}

Instead of creating an M-file, you may define the function with the \texttt{inline} command as follows:

\begin{verbatim}
f = inline('exp(-x^2)', 'x')
\end{verbatim}

and then proceed to calculate the integral with the \texttt{quad} command:

\begin{verbatim}
quad('f', 0, 1, .000001)
\end{verbatim}

The command \texttt{quad8} is similar to \texttt{quad} but uses a method with a higher order of accuracy.

\section{Solving differential equations using Matlab}


Consider the following initial value problem

\[ y' = f(t, y), \quad a \leq t \leq b \quad y(a) = \alpha. \quad (59) \]

\subsection{Symbolic Solutions}

Sometimes is possible to find an explicit solution (symbolic solution) of the differential equation (59). The command to find symbolic solutions is \texttt{dsolve}. It operates as follows. At the Matlab prompt \texttt{<<}, we type

\begin{verbatim}
dsolve('Dy = f(x, y)', y(x0) = y0', 'x')
\end{verbatim}

\textbf{Example 1.} To solve the initial value problem \( \frac{dy}{dx} = y - x, \quad y(0) = 2 \), we type
sol1 = dsolve('Dy = y - 4 * x', y(0) = 2' , x').

The output is

ans1 = 4 * x + 4 - 2 * exp(x)

Next, we want to plot the solution on the interval [0,3]. We take the step size equals to .1:

x = 0 : .1 : 3

If we type plot(x, ans1) we will get an error message. To be able to plot the solution, we need to make two modifications to the answer. First of all, the object to be plotted has to be a function (the command inline will do the job), and secondly it has to be a vector, i.e., the function has to be calculated at all values of x (0, .1, .2, .3, .4, ...3). The command vectorize will transform the scalar function into a vector one. Type:

\[ y1 = \text{inline}(\text{vectorize}(\text{ans1}), 'x') \]

Typing

plot(x, y1(x)) will produce the plot.

**Example 2.** Suppose that we want to solve the previous problem for arbitrary initial data, i.e., for \( y(0) = c \).

Let us run the \texttt{dsolve} command with the new initial data:

\[
sol2 = \text{dsolve}('Dy = y - 4 * x', y(0) = c', x').
\]

We get \( \text{sol2} = 4 * x + 4 + \exp(x) * (-4 + c) \).

We need to write the answer as a function (of \( x \) and \( c \)):

\[
y2 = \text{inline}(\text{vectorize}(\text{sol2}), 'x', 'c')
\]

To plot the family of curves, type:

axes; hold on
x=0:.1:3 for c=-1:4
plot(x, y2(x, c)) end axis tight title 'Solutions of equation 2' xlabel x ylabel y hold off

Notice that we have used some graphics options to improve the picture. Use the help command to find out about the above graph commands.

The previous examples deal with solving first order equations. Let us now use the \texttt{dsolve} command for higher order equations and systems. The notation for the first order derivative, \( y' \) is \( \text{Dy} \), for the second order derivative \( y'' \) is \( \text{D2y} \), and proceed likewise to represent higher order ones.

**Example 3.** Consider the differential equation

\[
y'' + y' - 6y = 20e^x.
\]
Let us run the \texttt{dsolve} command to solve it:

\begin{verbatim}
ode3='D2y+Dy -6*y= 20*exp(x)';
dsolve(node3, 'x')
\end{verbatim}

Matlab returns with
\begin{verbatim}
ans= -5*exp(x)+c1*exp(-3*x)+c2*exp(x)
\end{verbatim}

Note that the general solution of the equation depends on two arbitrary constants \(c_1\) and \(c_2\). To solve the differential equation with initial conditions \(y(0) = 0\) and \(y'(0) = 1\), we type
\begin{verbatim}
dsolve(node3, 'y(0)=0', 'y(1)=1', 'x')
\end{verbatim}
Matlab returns with
\begin{verbatim}
ans=-5*exp(x)+ 5*exp(2*x)
\end{verbatim}

7.2 Numerical solutions of differential equations

The numerical differential equation solver in Matlab is \texttt{ode45}.

7.2.1 First order equations

To solve a differential equation with Matlab, you first enter the function. If you are entering the data directly into the screen, you will type the command \texttt{inline} to define the function \(f\) of the right hand side of the equation. Let us look at the following example.

Example 1. Suppose you want to solve the initial value problem

\[
y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = .5
\]

At the Matlab prompt \texttt{>>}, you may type
\begin{verbatim}
f= inline('y - t^2 + 1', 't', 'y')
\end{verbatim}

Another method is to create a Matlab file, \texttt{example1.m}, as follows:

\begin{verbatim}
function z=example1(t, y);
    z = y - t^2 + 1;
\end{verbatim}

To plot the approximate solution on the interval \([0, 2]\), type
\begin{verbatim}
ode45(f, [0,2], .5)
\end{verbatim}

if \(f\) is an inline function, or
\begin{verbatim}
ode45('example1', [0,2], .5)
\end{verbatim}

if \texttt{example1} is a an M-file.

Now, suppose that you want to look at the plots of solutions for an interval of initial data, say, \(\alpha = .5\quad \alpha = .75t\alpha = 1.0, \ldots \quad \alpha = 2.0\). (Figure 2). Type:
\begin{verbatim}
ode45(f, [0,2], .5:.25:2);
\end{verbatim}

Note that \texttt{ode45} also solves nonlinear problems.

Example 2. Solve the following IVP:

\[
y' = t - \frac{e^{-t}}{y} + e^y, \quad t \in [0, 2], \quad y(0) = 1.
\]
7.2.2 First order systems

Example 3.
Solve the following initial value problem
\[
\begin{align*}
y_1' &= -3y_1 - y_2 \\
y_2' &= -y_1 - 3y_2, \quad y_1(0) = -1, y_2(0) = 1,
\end{align*}
\]
on the interval \(0 \leq x \leq 3\). We think of \(y_1\) and \(y_2\) as components of a vector \(y\). In Matlab, such components are written as \(y(1)\) and \(y(2)\). First, we define the (vector) function corresponding to the right hand side of the system (63):
\[
f = \text{inline}('[\text{-}3\ast y(1)\text{-}y(2); \text{-}y(1)\text{-}3\ast y(2)]', 'x', 'y')
\]
\[
[x, yanswer] = \text{ode45}(f, [0:.1:3], [-1 1]);
\]
If you do not type a semicolon, your screen will display a \(31 \times 1\) vector \(x\) and one \(31 \times 1\) vector for each of the solution components \(y(1)\) and \(y(2)\). These correspond to the numerical solution of the system.

Now, let us plot the solution. One useful graph is the one on the plane \((y(1), y(2))\). Type:
\[
\text{plot}(yanswer(:,1), yanswer(:,2))
\]
The syntaxes \(yanswer(:,1)\) corresponds to plotting the table of values (: notation) for the first component of the solution, and likewise \(yanswer(:,2)\) plots the values of the second component.

You may also plot each of the solution components as functions of \(x\):
\[
\text{plot}(x, yanswer(:,1)) \text{ and } \text{plot}(x, yanswer(:,2))
\]

Example 4.
Solve the following initial value problem:
\[
\begin{align*}
y_1' &= y_2 \\
y_2' &= -y_1 - y_2, \quad y_1(0) = 1, y_2(0) = 4,
\end{align*}
\]
on the interval \(0 \leq x \leq 20\).
\[
f = \text{inline}('[y(2); \text{-}y(1)\text{-}y(2)]', 'x', 'y')
\]
\[
[x, yanswer] = \text{ode45}(f, [0:.1:20], [1 4]);
\]
\[
\text{plot}(yanswer(:,1), yanswer(:,2))
\]

Example 5. Plot the solutions of system (64) for a collection of initial data,
\[
y_1(0) = a, \quad y_2(0) = b.
\]
We will take \(a = -6 : 6, b = -5 : 5\).
Type:
\[
\text{hold on}
\]
\[
\text{for } a=-6:6
\]
\[
\text{for } b=-5:5
\]

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Solutions of $\frac{dy_1}{dt}=y_2$, $\frac{dy_2}{dt}=-y_1-y_2$, $y_1(0)=a$, $y_2(0)=b$, $a=-6:6$, $b=-5:5$

```matlab
[x, yanswer]=ode45(f, [0:.1:20], [a b]);
plot(yanswer(:,1), yanswer(:,2))
end
end
hold off
```

### 7.3 Vector Fields

We now show a method to plot vector fields of differential equations (either first order, or autonomous second order ones); that is, the tangent vector to the (plane) solution curve.

**Example 1.** Consider the system

\[
\begin{align*}
y_1' &= \frac{1}{2}y_1 - 2y_2 \\
y_2' &= 2y_1.
\end{align*}
\]

We are going to plot the tangent vector to the solution path at every point of the plane (actually, in some portion of the plane only!). Let us call this vector $s = (s_1, s_2)$. In this example, $s_1 = \frac{1}{2}y_1 - 2y_2$ and $s_2 = 2y_1$. We achieve this with the `quiver` command.

Let us choose the portion of the plane for our plot, e.g. the square $(-2, 2) \times (-2, 2)$. The `meshgrid` command sets up the square. Type:

```matlab
[y1, y2]=meshgrid(-2:.2:2, -2:.2, 2)
s1=1/2*y1-2*y2
s2=2*y1
quiver(y1, y2, s1, s2, .5)
xlabel y1
ylabel y2
```
Now try to plot the vector fields for the problems of Homework 1. The ”.5” at the end of the quiver command reduces the length of the vectors by half. The purpose is cosmetic only.

The resulting plot is shown in the next picture.

7.4 Two dimensional systems: phase plane diagrams

We give examples on constructing the phase plane of two dimensional systems. We will proceed along the following steps: find the equilibrium (critical) points, write down the corresponding linearized systems, find the nature of the critical points and numerically construct the orbits.

Example 1.

\[
\begin{align*}
x' &= x - y, \\
y' &= 1 - x^2. 
\end{align*}
\]

(66)

Step 1. Find the critical points

Equilibrium points of the system are solutions of

\[
x - y = 0, \quad 1 - x^2 = 0.
\]

These are \((1, 1)\) and \((-1, -1)\).

Step 2. Linearized system about the critical points

Let us first consider \((1, 1)\) and define new variables,

\[
x = u + 1, \quad y = v + 1.
\]
The original system (omitting nonlinear terms) for the new variables, i.e., the linearized system is:

\[
\begin{bmatrix}
  u' \\
  v'
\end{bmatrix} = \begin{bmatrix}
  1 & -1 \\
  -2 & 0
\end{bmatrix} \begin{bmatrix}
  u \\
  v
\end{bmatrix}.
\]

Eigenvalues and eigenvectors, respectively, of the matrix are

\[\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \lambda_2 = -1, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.\]

The equilibrium point is a saddle point, and the eigenvectors denote the directions of the separatrix curves at (1,1).

We now consider the point \((-1, -1)\) and define the new variables \((p, q)\) as

\[x = p - 1, \quad y = q - 1.\]

The linearized system has the form,

\[
\begin{bmatrix}
  p' \\
  q'
\end{bmatrix} = \begin{bmatrix}
  1 & -1 \\
  2 & 0
\end{bmatrix} \begin{bmatrix}
  p \\
  q
\end{bmatrix}.
\]

Eigenvalues of the latter matrix are,

\[\lambda_1 = 1/2 \pm \sqrt{7}/2 \, i.\]

Therefore, \((-1, -1)\) is an unstable spiral point.

**Step 3. Plot the orbits of the original system.** You may type the following statements to Matlab.

```matlab
f=inline('[x(1) - x(2); 1 - x(1)^2]', 't', 'x');
tspan=[0:.035:.90];
for a=-1.5:.18:1.4; for b=-1.5:.18:1.6;
  hold on
  xzero=[a;b];
  [t, xans]=ode45(f, tspan, xzero);
  plot(xans(:,1), xans(:,2))
end
end
```

7.5 Long-Term Behavior of Solutions

One important issue about systems of differential equations is to understand what happens to all solutions, \(x(t)\), as \(t \to \infty\). We have defined the terms of stable, unstable and asymptotically stable equilibrium points of linear systems (page 180 of textbook). Such concepts apply also to nonlinear systems of dimension higher than two.
Solutions of the system

\[ x' = f(x), \]

are vector valued functions \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \). An equilibrium solutions \( x_0 \) satisfies \( f(x_0) = 0 \). To determine its stability, we write down the linearized system as follows. First, do the Taylor expansion of \( f \) about \( (x_0) \):

\[ f(x_0 + u) = f(x_0) + J(x_0)u + R(u), \]

where \( J \) is the Jacobian matrix of partial derivatives:

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n}
\end{bmatrix}
\]

\( R(u) \) is the remainder of the expansion. The linearized system about the equilibrium \( x_0 \) is:

\[ u' = Ju. \]

**Theorem 7.1.** Theorem Let \( x_0 \) and \( J \) be as previously defined. Suppose that the real part of every eigenvalue of \( J \) is negative. Then \( x_0 \) is an asymptotically stable equilibrium point. Suppose that \( J \) has at least one eigenvalue with positive real part. Then \( x_0 \) is an unstable equilibrium point.
7.6 Limit cycles

Consider the system

\[\begin{align*}
x' &= -y + x(1 - x^2 - y^2) \\
y' &= x + y(1 - x^2 - y^2).
\end{align*}\]

Express the system in polar coordinates and describe all solutions.

Let \(x = r \cos \theta, \ y = r \sin \theta\). Solving for \(r\) and \(\theta\) gives:

\[\begin{align*}
r^2 &= x^2 + y^2, \\
\tan \theta &= \frac{y}{x}.
\end{align*}\] (67) (68)

Let us calculate the time derivative of both sides of equation (67):

\[2rr' = 2(xx' + yy') = 2(x(-y + x(1 - x^2 - y^2)) + y(x + y(1 - x^2 - y^2))) = 2(x^2 + y^2)(1 - x^2 - y^2) = r^2(1 - r^2).\]

So, for \(r \neq 0\),

\[r' = r(1 - r^2).\] (69)

Likewise, we differentiate both sides of equation (68) with respect to \(t\):

\[\sec^2 \theta \theta' = \frac{d}{dt} \left( \frac{y}{x} \right) = \frac{(xy' - yx')}{x^2} = \frac{1}{x^2} [x(x + y(1 - x^2 - y^2)) - y(-y + x(1 - x^2 - y^2))] = \frac{(x^2 + y^2)}{x^2} = \frac{r^2}{x^2} = \sec^2 \theta.
\]

This gives

\[\theta' = 1,\] (70)

and therefore, \(\theta = t + C\).

Note that the original system reduces to two uncoupled equations (69) and (70). From the solution of (70), i.e. \(\theta = t + C\), we see that \(\theta\) is steadily increasing with speed 1. This means that all solution curves spiral around the origin in the counterclockwise direction.

Note that equation (69) has two equilibrium points, \(r = 0, 1\). Moreover, for \(r > 1, r' < 0\), and so \(r(t)\) decreases to \(r = 1\); for \(r < 1, r' > 0\) and \(r(t)\) increases to \(r = 1\). Thus \(r = 1\) is an asymptotically stable equilibrium point, and all other solution curves starting away from the origin spiral toward it. Hence the unit circle is the limit set of any point of \(\mathbb{R}^2\), except the origin.

The circle \(r = 1\) is a limit cycle; it attracts solutions starting in either side of the circle. Note that \(r = 1\) is an orbit of the problem. We can also say that the curve \(r = 1 (x^2 + y^2 = 1)\) is an invariant set of the equations. (Why?)
7.7 The Lorenz system

The model discovered by the mathematician and meteorologist Lorenz (1963) applies to atmospheric turbulence:

\[
\begin{align*}
x' &= -ax + ay, \\
y' &= rx - y - xz, \\
z' &= -bz + xy.
\end{align*}
\]

$a, r$ and $b$ are parameters. The equilibrium points of the system are:

\[
c_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad c^+ = \begin{bmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r - 1 \end{bmatrix}, \quad c^- = \begin{bmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r - 1 \end{bmatrix}.
\]

Note that if \( r < 1 \) there is only an equilibrium solution, and there are three for \( r > 1 \). The Jacobian matrix of the Lorenz system is:

\[
J(x, y, z) = \begin{bmatrix} -a & a & 0 \\ (r-z) & -1 & -x \\ y & x & -b \end{bmatrix},
\]

where \((x, y, z)\) denotes an equilibrium point. Let us consider the special set of parameters \( a = 10, b = 8/3, r = 28 \) and study the properties of the equilibrium points. The matrix

\[
J(0, 0, 0) = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}
\]

has eigenvalues \(-22.8277, 11.8277, -2.6667\). So, \((0, 0, 0)\) is unstable.
The matrix

$$J(6\sqrt{2}, 6\sqrt{2}, 27) = \begin{bmatrix}
-10 & 10 & 0 \\
1 & -1 & -6\sqrt{2} \\
6\sqrt{2} & 6\sqrt{2} & -8/3
\end{bmatrix}$$

has eigenvalues $-13.8546, 0.094 + 10.1945i, 0.094 - 10.1945i$. So, $(6\sqrt{2}, 6\sqrt{2}, 27)$ is unstable. Likewise,

$$J(-6\sqrt{2}, -6\sqrt{2}, 27) = \begin{bmatrix}
-10 & 10 & 0 \\
1 & -1 & 6\sqrt{2} \\
-6\sqrt{2} & -6\sqrt{2} & -8/3
\end{bmatrix}$$

has the same eigenvalues as the previous one and consequently the equilibrium point is unstable.

We now plot the solution corresponding to initial data $[-2, -1, 1]$ on the time interval $tspan = [0, 70]$, using 'ode45'. Observe the different graphs. (Notation: $x = y(1), y = y(2), z = y(3).$)
7.8 Invariant sets

An invariant set for a system $\mathbf{x}' = f(\mathbf{x})$ of dimension $n$ is a set $S \subset \mathbb{R}^n$ with the property that if $\mathbf{x}(t)$ is a solution to the system with its initial value $\mathbf{x}(0) \in S$, then $\mathbf{x}(t) \in S$ for all $t \geq 0$.

**Example** Consider the system

$$x' = (1 - x - y)x, \quad y' = (4 - 7x - 3y)y.$$ 

- Find and classify critical points;
- Show that the $x$-axis and the $y$-axis are invariant sets;
- Show that the positive quadrant is an invariant set;
- Show that the square $x \in (0, 1), y \in (0, 1.5)$ is an invariant set;
- Find all the invariant sets inside the square;
- Discuss and justify the behaviors of all the trajectories of the system.

1. There are four equilibrium points:
   - $(0, 0)$ is a nodal source;
   - $(\frac{1}{4}, \frac{3}{4})$ is a saddle point;
   - $(0, \frac{4}{3})$ and $(1, 0)$ are nodal sinks.

2. Consider initial data $(x_0, 0)$. Let $(\bar{x}(t), \bar{y}(t))$ be a solution of the system $x' = (1-x)x, \quad y' = 0$. Note that $\bar{x}$ is a function satisfying $x' = (1-x)x$ and $\bar{y}(t) = 0$. It is easy to check that $(\bar{x}(t), 0)$ also solves the original system, for initial data $(x_0, 0)$. Moreover, $(\bar{x}(t), 0)$ is the unique solution to such a system for the given initial data. Consequently, any solution with $y(0)=0$, satisfies $y(t) = 0$ for all $t > 0$. Thus, the line $y = 0$, the $x$-axis, is an invariant set. Likewise, we show invariance of the $y$-axis.
3. The positive quadrant is invariant. Notice that the only way for solutions starting in the positive quadrant to leave the region is by crossing one of the axis. However, a trajectory reaching a point in the $x$-axis (or the $y$-axis) has to continue on the line due to the invariance shown in part 2.

4. First of all, let us show that a trajectory starting inside the given square and meeting the line $x = 1$ cannot cross it. For this, let us set $x = 1$ in the governing system: $x' = -xy$ and $y' = -3(1 + y)y$. Note that $x' < 0$, $y' < 0$ at points $(1, y)$, i.e., the vector field (tangent vector to the trajectory) points SW. So, since the trajectory turns SW, it cannot leave the square. Likewise, we argue about how trajectories cannot cross the line $y = 1.5$.

5. To determine all invariant sets, let us find the line where $x' = 0$ and the line where $y' = 0$ (these are often called nullclines of the system).

Setting $x' = 0$ in the first equation gives us: $x = 0$ or $x + y = 1$. In part 2, we have shown that $x = 0$ is invariant (and therefore trajectories cannot cross it). We only need to examine the line $x + y = 1$. Let us substitute $x = 1 - y$ in the second equation; this gives

$$y' = (-3 + 4y)y.$$ 

Therefore,

- $x' = 0$ and $y' < 0$ for $y < 3/4$ (on the line $x + y = 1$): the vector field is vertical and pointing downwards;
- $x' = 0$ and $y' > 0$ for $y > 3/4$ (on the line $x + y = 1$): the vector field is vertical and pointing upwards.

Setting $y' = 0$ in the second equation gives us: $y = 0$ or $7x + 3y = 4$. In part 2, we have shown that $y = 0$ is invariant (and therefore trajectories cannot cross it). We only need to examine the line $7x + 3y = 4$. Let us substitute $y = \frac{1}{3}(4 - 7x)$ in the first equation; this gives

$$x' = -\frac{1}{3}(-1 + 4x).$$

Therefore,

- $y' = 0$ and $x' > 0$ for $x > \frac{1}{4}$ (on the line $7x + 3y = 4$); the vector field is horizontal and points towards the right);
- $y' = 0$ and $x' < 0$ for $x < \frac{1}{4}$ (on the line $7x + 3y = 4$); the vector field is horizontal and points towards the left).

The information gathered so far is shown in the next figure (equilibrium points are marked with circles). Note that the arrows in the nullcline graphs show invariant regions I, II, III and IV. 6. Solutions corresponding to initial data in region II, in addition to remaining there for all time, also satisfy, $x(t) \to 1$ and $y(t) \to 0$, as $t \to \infty$. This is due to the fact the the vector field points SE for all time. Once the trajectory gets sufficiently close to the equilibrium point, the fact that it is asymptotically stable (nodal sink), cause the trajectory to tend to it as $t$ grows. Likewise, we show that trajectories with initial data in region IV will tend to $(0, 4/3)$ as $t \to \infty$.

Similarly, a solution with initial data in region I will approach $(0, 4/3)$ or $(1, 0)$ or will enter region II or region IV. In either case, trajectories will tend to either $(1, 0)$ or $(0, 4/3)$ as $t \to \infty$. 

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Nullclines of $\frac{dx}{dt} = (1-x-y)x$, $\frac{dy}{dt} = (4-7x-3y)y$

$x+y=1$

$3y + 7x = 4$
7.9 Energy methods

Finding an energy equation provides a method for the rigorous justification of the phase plane diagram of a two-dimensional autonomous system. Furthermore, it is also applicable to systems of many dimensions. To illustrate the main points of such a method, we will begin with a very well known example:

The linear oscillator.

\[ m y'' + ky = 0. \]

Letting \( u = y, \ u' = v \), we rewrite it as

\[ u' = v, \quad v' = -\frac{k}{m}u. \]

Denote \( a^2 = \frac{k}{m} \), and calculate

\[ \frac{dv}{du} = \frac{dv}{dt} \frac{dt}{du}. \]

Using the given equations,

\[ \frac{dv}{du} = -\frac{a^2u}{v}, \quad vdv = -a^2udu. \]

Integrating the latter yields,

\[ \frac{1}{2}v^2 + \frac{a^2}{2}u^2 = E, \tag{76} \]

where \( E \) is an arbitrary constant. Identifying the first term on the left with the kinetic energy of the system and the second one with the potential energy, i.e.,

\[ U(u) = \frac{a^2}{2}u^2, \]

equation (76) establishes conservation of energy: the sum of kinetic plus potential energy is constant.

The collection of curves in the \( u - v \) plane (phase plane) corresponding to different values of \( E \) give the trajectories or orbits of the system. Solutions of the given system of differential equations correspond to orbits in the phase plane.

Let us show how to use the orbits that we found to obtain solutions of the system.

Suppose that we want to calculate the solution corresponding to initial values \( u(0) = u_0 \) and \( v(0) = v_0 \). Substituting these values in equation \( E \), it singles out the special value, \( E_0 \), whose orbit corresponds to the solution that we are looking for. For such an \( E_0 \), we calculate,

\[ v = \pm \sqrt{2(E_0 - U(u))}. \tag{77} \]

Properties of the solutions:

- Note that (77) is symmetric with respect to the \( u \)-axis; (these is the case for all orbits of the system);
Observe the $v = 0$ at those points $u$ such that $U(u) = E$. In such points, the speed $v = u'$ changes sign, turning from positive to negative (or vice versa); at such point, the displacement $u$ reaches each maximum (or minimum), and begins to decrease (or increase), up to reaching the minimum value (or maximum) where $v = 0$ again. This process keeps repeating itself, and therefore, showing the periodic nature of the solutions.

In addition to finding the energy equation (76) to calculate orbits, we also need the study of the critical points, in order to justify the phase plane structure of the problem. In this case, we see that $(0, 0)$ is the only equilibrium point. You can easily check that the linearized system about $(0, 0)$, or equivalently, the Jacobian matrix of the system has purely imaginary eigenvalues; i.e., $(0, 0)$ is a center.

Now, you can use Matlab to construct the phase plane diagram and check the results previously obtained.

The equations of motion of the pendulum of mass $m$ attached to a solid rod of length $L$ are given by:

$$T - mg \cos \theta = 0,$$

$$mL\theta'' = -mg \sin \theta. \quad (78)$$

The first equation expresses the balance of the weight component along the direction of the rod by the support, and the second one corresponds to Newton’s law of motion due to the transverse component of the weight. Letting $\omega = \theta'$, we can express the second equation as a system of two first order ones:

$$\theta' = \omega, \quad \omega' = -a \sin \theta, \quad (79)$$

where $a = \frac{g}{L}$.

In order to study all solutions of the (79) we study the equilibrium points and find the energy equation. Observe that the equilibrium points of the system satisfy

$$\omega = 0, \quad \text{and} \quad \sin \theta = 0.$$
The latter gives
\[ \theta = \pi n, \quad n = 0, \pm 1, \pm 2, \ldots \] (80)

The Jacobian matrix of the system is:
\[
\begin{bmatrix}
0 & 1 \\
-a \cos \theta & 0
\end{bmatrix}
\]

We can distinguish two types of equilibrium points. In the case that \( \theta = \pi n, n = 0, \pm 2, \pm 4, \ldots \), \( \cos \theta = 1 \), and
\[
J = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}.
\]

Eigenvalues of \( J \) are \( \lambda = \pm i \sqrt{a} \). Therefore, points \((0, \pi n), n = 0, \pm 2, \ldots \) are centers.

Likewise, we find (check the Jacobian) that eigenvalues of \( J \) in the case that \( \theta = n\pi, n = \pm 1, \pm 3, \ldots \) are saddle points.

Next, we obtain the energy equation following the same procedure as for the harmonic oscillator of the previous example. It is given by,
\[
\frac{1}{2} \omega^2 + U(\theta) = E, \quad (81)
\]

where \( U(\theta) = -a \cos \theta \). Let us plot \( U(\theta) \) for the pendulum.

Solving (81) for \( \omega \) yields,
\[
\omega = \pm \sqrt{2(E - U(\theta))} \quad (82)
\]

We now have enough information to obtain all the orbits of the system. For this, we analyze (82) for the following choices of energy constant:

- \( E = E_0 = -a \);
- \( E = E_1; -a < E_1 < a \);
- \( E = E_2 = a \);
• \( E > a. \)

The only solutions that exist in case 1 are those corresponding to the stable critical points (centers). No solution exist for initial data other than \((n\pi, 0)\), with \(n\) even. For initial data as in case 2, closed orbits around the centers can be found. These correspond to periodic solutions of the system. Orbits corresponding to \(E_2 = a\) are separatrices of the saddle points; these enclose periodic orbits. Solutions with initial data such that \(E > a\) have always nonzero velocity \(\omega\), therefore \(\theta\) is monotonic and the corresponding orbits are unbounded.

As an exercise, use Matlab to draw the phase diagram for this problem.

### 7.9.1 Conservation of energy

The techniques described in the previous examples are valid for systems of the form

\[
x' = y, \quad y' = f(x),
\]

where \(f\) is a continuous function on its domain of definition. Let us calculate,

\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{f(x)}{y},
\]

and consequently, \(ydy = f(x)dx\), yielding

\[
\frac{y^2}{2} + U(x) = E, \quad \text{where } U(x) = -\int f(x)\,dx.
\]

Equation (84) gives the orbits of the system. Moreover, the properties of the critical points of \(U(x)\) determine the nature of the equilibrium solutions of the equation. In fact, the Jacobian of the system (83) is

\[
J = \begin{bmatrix} 0 & 1 \\ f'(x) & 0 \end{bmatrix},
\]

with eigenvalues \(\lambda = \pm \sqrt{f'(x)}\). where \(x\) corresponds to an equilibrium value. Note that

- \(U'(x) = -f(x)\): equilibrium points satisfy \(U'(x) = 0 = f(x)\);

- \(U''(x) = -f'(x)\): a maximum of \(U\) corresponds to \(f'(x) > 0\), and therefore the eigenvalues of \(J\) are real (with opposite sign), in which case \((x, 0)\) is a saddle point; likewise a minimum of \(U\) at \(x\) corresponds to a center of the system.

### 7.10 The Method of Liapunov

Now, consider a more general autonomous system

\[
x' = f(x, y), \quad y' = g(x, y).
\]

Suppose that \((x_0, y_0)\) is an equilibrium point of the system. This method consists in finding a continuously differentiable function \(V(x, y)\) with the following properties:
\( V(x_0, y_0) = 0; \) moreover there is an interval \( \mathcal{U} \) of \((x_0, y_0)\) such that \( V(x, y) \geq 0 \), in \( \mathcal{U} \), and \( V(x, y) > 0 \) for \((x, y) \neq (x_0, y_0);\)

- On solutions \((x(t), y(t))\) of the system, \( V \) is a function of \( t \), i.e., \( V(t) = V(x(t), y(t)) \), such that \( \frac{dV}{dt} \leq 0. \)

A function \( V(x, y) \) with Property 1 is called a **positive definite** function. Likewise a function satisfying Property 2 is called **negative semidefinite** (note that we are not requiring \( \frac{d}{dt}V(x(t), y(t)) < 0 \), i.e., strictly). Let us explore Property 2. By the chain rule and using (85), we find that

\[
\frac{d}{dt}V(x(t), y(t)) = \nabla V(x(t), y(t)) \cdot (f(x(t), y(t)), g(x(t), y(t))).
\] (86)

A system of differential equations having a function \( V(x, y) \) with Properties 1 and 2 is said to have a Liapunov function. The following theorem states how the existence of a Liapunov function \( V(x, y) \) helps determining the stability properties of the equilibrium solutions.

**Theorem (Liapunov).** Suppose that \((x_0, y_0)\) is an equilibrium solution of (85). Suppose that there is a continuously differentiable function \( V \) defined on a neighborhood \( \mathcal{U} \) of \((x_0, y_0)\) that is positive definite with a minimum at \((x_0, y_0)\). Then

- If \( \frac{d}{dt}V \) is negative semidefinite in \( \mathcal{U} \) then \((x_0, y_0)\) is a stable equilibrium point.
- If \( \frac{d}{dt}V \) is negative definite in \( \mathcal{U} \) then \((x_0, y_0)\) is an asymptotically stable equilibrium point.

This result is valid for any system dimension, not necessarily a plane system only.

**Exercise 1.** Consider the damped harmonic oscillator, 

\[
my'' + \mu y' + ky = 0
\]

where \( m, \mu \) and \( k \) are positive (nonzero) constants. Show that the energy \( V(x, y) = \frac{m}{2} y^2 + \frac{k}{2} y^2 \) is a Liapunov function of the system. Use it to determine the type of stability of the equilibrium point \((0, 0)\).

**Exercise 2.** Consider the equation of the damped nonlinear pendulum,

\[
ml\theta'' = -mg \sin \theta - \mu \theta',
\]

where \( m, L, g, \mu \) are positive (nonzero) constants.

- Find a Liapunov function for the system.
- Use the Liapunov function to study the stability of all equilibrium solutions.