1 Math 8401: Assignment 1

1.1 Part I. Scaling and non-dimensionalization

1. In a classical work modeling the outbreak of the spruce bud worm in Canada’s balsam fir forests, researchers proposed that the bud worm population \( n = n(t) \) was governed by the law

\[
\frac{dn}{dt} = rn(1 - \frac{n}{K}) - P(n),
\]

where \( r \) and \( K \) are the growth rate and carrying capacity, respectively, and \( P(n) \) is a bird predation term given by

\[
P(n) = \frac{bn^2}{a^2 + n^2},
\]

where \( a \) and \( b \) are positive constants.

1. Determine the dimensions of the constants \( a \) and \( b \).

2. Graph the predation rate \( \frac{P(n)}{b} \) for \( a = 1, 5, 10 \) and make a qualitative statement about the effect that the parameter \( a \) has on the model.

3. Select dimensionless variables \( N = \frac{n}{a} \) and \( \tau = \frac{bt}{a} \) and reduce the differential equation to dimensionless form.

4. Find the equilibrium solutions of the dimensionless equation. (There are multiple solutions for selected parameter values.)

Solution.

1. \([a]=\text{population}, \ [b]=\ [P]=\text{population/time}, \ [r]=\text{time}^{-1} .\]

2. Plot \( f(n) = \frac{n^2}{a^2 + n^2} \). Increase of the parameter \( a \) translates into a decrease of the bird predatory rate.

3. Substituting the giving dimensionless variables \( N \) and \( \tau \) into the ordinary differential equation gives

\[
\frac{dN}{d\tau} = \alpha N(1 - \frac{a}{K}N) - \frac{N^2}{1 + N^2}, \quad \alpha := \frac{ar}{b}.
\]

4. The equilibrium solutions satisfy the equation

\[
\alpha N(1 - \frac{a}{K}N) - \frac{N^2}{1 + N^2} = 0.
\]

Thus, \( N = 0 \) is a solution as well as the roots of the polynomial

\[
f(N) := N^3 - \gamma N^2 + (\beta^{-1} + 1)N - \gamma = 0, \quad \gamma := \frac{K}{a}.
\]

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\(^1\) Problem 6, page 31, Applied Mathematics, J.D. Logan, Wiley 2006.
Figure 1: The left figure represents $P(n)$. The figure on the right illustrates the nonzero equilibrium points of the ordinary differential equation, and, in particular, how the number of solutions changes with parameter values: $\gamma$ and $\beta$. Increase in $\gamma$ shows an increase of the number of roots of the polynomial $f(N)$. Also, we observe that as long as inequality (1.1) holds, the graph of $f$ shows very low sensitivity to the parameter $\beta$, for $\beta < 0.1$.

Note that this equation has at least one positive root $N_0 > 0$. A necessary condition for the equation to have multiple roots (2 or 3) is that

$$\gamma^2 \geq 3(1 + \beta^{-1}),$$

hold. The latter condition ensures the existence of a maximum and minimum of $f(N)$.

The adjoint plots illustrate how the number of solutions changes with parameter values: $\gamma$ and $\beta$. Increase in $\gamma$ shows an increase of the number of roots of the polynomial $f(N)$. Also, we observe that as long as inequality (1.1) holds, the graph of $f$ shows very low sensitivity to the parameter $\beta$, for $\beta < 0.1$.

Note. Other methods to resolve part (4) of the problem can be used. For instance, studying the tangent lines to the graph of the function on the right hand side of (2).

2. The length $L$ of an organism depends upon time $t$, its density $\rho$, its resource assimilation rate $a$ (mass per unit area and per unit time), and its resource use rate $b$ (mass per volume per time). Show that there is a physical law involving two dimensionless quantities only.

Solution. The law governing the growth of the organism is of the form

$$f(L, T, \rho, a, b) = 0.$$

We look for exponents $\alpha, \beta, \gamma, \delta, \kappa$ such that

$$L^\alpha T^\beta \rho^\gamma a^\delta b^\kappa = 1.$$

The dimensional form of this relation beomes

$$L^\alpha T^\beta \left(\frac{M^\gamma}{L^{3\gamma}}\right)^\delta \left(\frac{M}{L^3 T}\right)^\kappa = 1.$$
Hence, after some straightforward calculations, we have

\[ \alpha = \delta, \]
\[ \beta = \delta + \kappa, \]
\[ \gamma = -\delta - \kappa. \]

Equivalently, the previous relations in matrix form become

\[
\begin{pmatrix}
1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & -1 \\
0 & 0 & 1 & -1 & -1
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta \\
\kappa
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Note that the rank of the matrix of the coefficients is three (the size of the largest nonsingular matrix). According to the theory of linear systems, we know that the collection of unknowns \([\alpha \ \beta \ \gamma \ \delta \ \kappa]^T\) can be expressed in terms of two independent ones (number of unknowns - matrix rank). That is, the set of solutions forms a vector space of dimension 2. Hence, there are only two independent parameter groups. For instance, two of such groups are

\[ \frac{T}{\rho} \quad \text{and} \quad \frac{LT\alpha}{\rho}. \]


**Solution.**

1. A physical law for such pendulum is of the form

\[ f(p, l, g, m) = g(\theta_0), \]

with \( p, \ l, \ m \) and \( g \) denoting the period, length and mass of the pendulum, and the gravity constant, respectively, for some functions \( f \) and \( g \). (Note that, since \( \theta_0 \), an angle, it is dimensionless.) We then require that

\[ T^\alpha L^\beta (\frac{L}{T^2})^\gamma M^\delta = 1, \]

which yields a law of the form

\[ \frac{p^2 g}{L} = c(\theta_0). \]

That is, we recover the well known law that

\[ p = c(\theta_0) \sqrt{\frac{L}{g}}. \]

In particular, note that the period is independent of the mass of the pendulum. The latter expression, with \( c = 1 \), is exact for the liner pendulum.
2. Suppose that the pendulum on our desk has length $L_0$. In our experiment, we measure its period, $p_0$, which satisfies the relation

$$p_0 = c(\theta_0) \sqrt{\frac{L_0}{g}}.$$  

This gives

$$c(\theta_0) = p_0 \sqrt{\frac{g}{L_0}}.$$  

To calculate the period $p_1$ of the largest pendulum ever built, we then need to evaluate

$$p_1 = p_0 \sqrt{\frac{L_1}{L_0}},$$

where $L_1$ denotes its length.

3.  

$$p = c\theta_0 \sqrt{\frac{L}{g}},$$

where $c$ is a constant.


**Solution.** From the given force relation, we derive that the dimensions of the gravitational constant $G$ are

$$[G] = \frac{L^3}{T^2 M}.$$  

The law of luminosity is given by

$$f(p, r, m, G) = 0,$$

where $p$, $r$ and $m$ denote the period, average radius and mass of the star, respectively. $f$ is an unknown function. A dimensional analysis of the previous relations gives

$$T^\alpha L^\beta M^\gamma \left( \frac{L^3}{T^2 m} \right)^\delta = 1,$$

for exponents $\alpha$, $\beta$, $\gamma$ and $\delta$ to be determined. The usual argument gives

$$\alpha = 2\delta, \quad \beta = -3\delta, \quad \gamma = \delta.$$  

From these, we immediately derive that

$$p = C \sqrt{\frac{r^3}{G}},$$

where $C$ is an unknown constant.

5. Consider the partial differential equation

\[
\frac{\partial u}{\partial t} = \kappa \Delta u + au \quad \text{for} \quad x \in \Omega \subset \mathbb{R}^3,
\]

\[u(x,0) = u_0(x) \quad \text{in} \quad \Omega,
\]

\[u = 0 \quad \text{on} \quad \partial \Omega
\]

where \( \Omega \) is open and bounded.

- Write down the energy law of the problem.
- Find the long-time behavior of the solutions, that is, the limit \( u(x,t) \) as \( t \to \infty \).

We proceed as in the notes to arrive at the inequality

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx = -\kappa \int_{\Omega} |\nabla u|^2 \, dx + a \int_{\Omega} u^2 \, dx.
\]  

(3)

If, in addition, \( a \leq 0 \), it follows that

\[
\int_{\Omega} u^2(x,t) \, dx \leq \int_{\Omega} u^2(x,0) \, dx, \quad t \geq 0.
\]  

(4)

In order to get a sharper result, we appeal to the following change of variable

\[v(x,t) = e^{-at}u(x,t).
\]

A simple calculation shows that \( v_t = (au + u_t)e^{-at}, \Delta v = e^{-at}\Delta u \) and \( v(x,0) = u(x,0) \). Substituting these expressions into equation (3), we find that \( v \) satisfies the equation

\[v_t(x,t) = \kappa \Delta v(x,t).
\]

Hence

\[
\int_{\Omega} v^2(x,t) \, dx \leq \int_{\Omega} v^2(x,0) \, dx, \quad t \geq 0,
\]

which, in terms of the original function \( u \) becomes

\[e^{-2at} \int_{\Omega} u^2(x,t) \, dx \leq \int_{\Omega} u^2(x,0) \, dx, \quad t \geq 0.
\]

Hence

\[
\int_{\Omega} u^2(x,t) \, dx \leq e^{2at} \int_{\Omega} u^2(x,0) \, dx, \quad t \geq 0.
\]

Taking limits as \( t \to \infty \) on both sides of the inequality, we get

\[
\lim_{t \to \infty} \int_{\Omega} u^2(x,t) \, dx = 0, \quad t \geq 0.
\]

By continuity of \( u \) and the localization theorem, we conclude that \( \lim_{t \to \infty} u(x,t) = 0, \forall x \in \Omega \).