1. Let $GL_2(\mathbb{Z}_2)$ be the group of invertible $2 \times 2$ matrices with entries in $\mathbb{Z}_2$. List its elements. What is the order of the group? Find all of its subgroups.

2. a. List all of the generators of $\mathbb{Z}_{20}$.
   
   b. List the elements of the subgroups $\langle 3 \rangle$ and $\langle 7 \rangle$ in $U(20) = \mathbb{Z}_{20}^\times$.
   
   c. Find all subgroups of $\mathbb{Z}_{18}$ and $U(11) = \mathbb{Z}_{11}^\times$.

3. a. Let $a$ be an element in a group. If $|a| = n$, show that $\langle a^k \rangle = \langle a^{\text{gcd}(n,k)} \rangle$.
   
   b. Let $a$ be an element in a group. Suppose that $|a| = 24$. Find a generator of $\langle a^{21} \rangle \cap \langle a^{10} \rangle$.

4. Given the permutations $\sigma = (1\ 2\ 4), \tau = (1\ 3)(2\ 4) \in S_4$, compute the following elements:
   
   a. $\sigma^{-1}$
   
   b. $\sigma \tau$
   
   c. $\tau \sigma$
   
   d. $\sigma^2$
   
   e. $\sigma^2 \tau$
   
   f. $\sigma \tau \sigma^{-1}$
   
   g. $\tau \sigma \tau^{-1}$

5. a. Prove that a $k$-cycle in $S_n$ is an element of order $k$.
   
   b. Prove that when we represent a permutation as a product of disjoint cycles, the order of the permutation is the least common multiple of the lengths of these cycles.

6. Determine if $\sigma = (1\ 2)(1\ 3\ 4)(1\ 5\ 2), \tau = (1\ 2\ 4\ 3)(3\ 5\ 2\ 1) \in S_5$ are even or odd.

7. Prove that $A_n$ contains an $n$-cycle if and only if $n$ is odd.

Through the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.

Signed: ________________________________