Name: _____

Problem Set 10
Math 4281, Fall 2016
Due: Wednesday, November 16

Complete the following items, staple this page to the front of your work, and turn in your assignment at the beginning of class on the due date. Remember to list your collaborators on each problem.

Ring isomorphisms

1. a. Prove that the composition of two ring isomorphisms is a ring isomorphism.
   b. Suppose that $\phi\colon R \to S$ is a ring isomorphism. Prove that the inverse function $\phi^{-1}\colon S \to R$ is a ring homomorphism (and therefore also an isomorphism).

2. Let F be a field, $f(x) \in F[x]$, and K be a field extension of F containing the root α of $f(x)$.
   a. If $\sigma\colon K \to K$ is a ring isomorphism with the property that $\sigma(a) = a$ for all $a \in F$, show that $\sigma(\alpha)$ is likewise a root of $f(x)$.
   b. Apply (a) to show that the complex roots of a real polynomial occur in conjugate pairs.
   c. Apply (a) to show that if $n \in \mathbb{N}$ is not a perfect square, and $\sqrt{n}$ is a root of $f(x) \in \mathbb{Q}[x]$, then $-\sqrt{n}$ is a root as well.

Vector spaces and field extensions

3. Prove that the real numbers 1 and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$. Do the same for $1, \sqrt{3}$, and $\sqrt{5}$.

4. Give a basis for each of the given vector spaces over the given field. What is the degree of each field extension?
   a. $\mathbb{Q}(\sqrt{3}, i)$ over $\mathbb{Q}$
   b. $\mathbb{Q}(\sqrt{3}, i)$ over $\mathbb{Q}(i\sqrt{3})$
   c. $\mathbb{Z}_2[x]/\langle x^3 + x + 1\rangle$ over $\mathbb{Z}_2$
   d. $\mathbb{Q}(\sqrt[5]{8})$ over $\mathbb{Q}$

5. Let F be a field. Suppose that K is a field extension of F of finite degree. Prove that if $\alpha \in K$, then there is an irreducible polynomial $f(x) \in F[x]$ having α as a root. (Hint: If $[K : F] = n$, consider $1, \alpha, \alpha^2, \ldots, \alpha^n$.)

Throughout the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.
Signed: _____