Name: _____

Problem Set 8
Math 4281, Fall 2016
Due: Wednesday, November 2

Complete the following items, staple this page to the front of your work, and turn in your assignment at the beginning of class on the due date. Remember to list your collaborators on each problem.

Ring homomorphisms and ideals

1. Find all ideals in $\mathbb{Z}$ and in $\mathbb{Z}_6$.

2. Let R be a commutative ring with 1, and let $a_1, \ldots, a_n \in R$. Show that

$$\langle a_1, \ldots, a_n \rangle := \{r_1 a_1 + \cdots + r_n a_n \mid r_i \in R \text{ for all } i\} \subseteq R$$

is an ideal in R.

3. Let R be a commutative ring with 1, and let $I, J \subset R$ be ideals. Define

$$I \cap J = \{a \in R \mid a \in I \text{ and } a \in J\} \quad \text{and} \quad I + J = \{a + b \in R \mid a \in I, b \in J\}.$$

   a. Prove that $I \cap J$ and $I + J$ are ideals.
   b. Suppose $R = \mathbb{Z}$ or $F[x]$ for a field F, $I = \langle a \rangle$, and $J = \langle b \rangle$. Identify $I \cap J$ and $I + J$ in terms of a and b.
   c. Let $a_1, \ldots, a_n \in R$. Prove that $\langle a_1, \ldots, a_n \rangle = \langle a_1 \rangle + \cdots + \langle a_n \rangle$.

4. Let R be a commutative ring with 1.

   a. Prove that if $I \subseteq R$ is an ideal and $1 \in I$, then $I = R$.
   b. Prove that $a \in R$ is a unit if and only if $\langle a \rangle = R$.
   c. Prove that the only ideals in R are $\langle 0 \rangle$ and R if and only if R is a field.

Quotient rings

5. Prove that if F is a field and $f(x) \in F[x]$ is not irreducible, then $F[x]/\langle f(x) \rangle$ contains zero divisors.

6. Give the addition and multiplication tables of $\mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle$.

Throughout the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.
Signed: _____