Submit solutions to three of the following problems.

1. Determine whether \( f = xy^3 - z^2 + y^5 - z^3 \) is in the ideal \( I = \langle -x^3 + y, x^2y - z \rangle \).

2. Assume that \( k \) is an algebraically closed field. Identify \( \mathbb{A}^9(k) \) with the space of \((3 \times 3)\)-matrices \( A = [a_{i,j}] \). Let \( \rho: \mathbb{A}^9(k) \to \mathbb{A}^9(k) \) be the rational map defined by
   \[
   A \mapsto A \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} A^{-1}.
   \]

   (a) Find equations for the smallest affine variety \( X \) containing the image of \( \rho \).
   (b) Show that \( X \) is the set of all nilpotent \((3 \times 3)\)-matrices.

3. Use the method of Lagrange multipliers to find the point(s) on the surface defined by \( x^4 + y^2 + z^2 - 1 = 0 \) that are closest to the point \((1,1,1)\) in \( \mathbb{R}^3 \).
   **Hint:** Proceed as in Example 3 in §2.8.

4. Suppose that \( k \) is a field and \( \varphi: k[x_1, \ldots, x_n] \to k[x_1] \) is a ring homomorphism that is the identity on \( k \) and maps \( x_1 \) to \( x_1 \). Given an ideal \( I \subseteq k[x_1, \ldots, x_n] \), prove that \( \varphi(I) \subseteq k[x_1] \) is an ideal.
   **Hint:** In the proof of Theorem 3.5.2, we use this result when \( \varphi \) is the map that evaluates \( x_i \) at \( a_i \) for \( 2 \leq i \leq n \).

5. Consider the ideal \( I = \langle x^2y + xz + 1, xy - xz^2 + z - 1 \rangle \) discussed in §3.5.
   (a) Show that the partial solution \((b, c) = (0, 0)\) does not extend to a solution \((a, 0, 0) \in V(I)\).
   (b) In the text, it is shown that \( g_o = g_1 \) for the partial solution \((1, 1)\). Show that \( g_o = g_3 \) works for all partial solutions different from \((1, 1)\) and \((0, 0)\).