Submit solutions to three of the following problems.

1. Let $I$ and $J$ be ideals in $S = \mathbb{k}[x_1, \ldots, x_n]$, where $\mathbb{k}$ is an arbitrary field.
   (a) If $I^m \subseteq J$ for some integer $m > 0$, then show that $\sqrt{I} \subseteq \sqrt{J}$.
   (b) Prove that $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

2. Two ideals $I$ and $J$ in $S := \mathbb{k}[x_1, \ldots, x_n]$ are comaximal if and only if $I + J = S$.
   (a) If $\mathbb{k}$ is an algebraically closed field, then show that $I$ and $J$ are comaximal if and only if $V(I) \cap V(J) = \emptyset$. Give an example to show that this is false in general.
   (b) If $I$ and $J$ are comaximal, then show that $IJ = I \cap J$.
   (c) If $I$ and $J$ are comaximal, then show that $I^i$ and $J^j$ are comaximal for all positive integers $i$ and $j$.

3. Let $I, J$ be ideals in $\mathbb{k}[x_1, \ldots, x_n]$ and suppose that $I \subseteq \sqrt{J}$. Show that $I^m \subseteq J$ for some integer $m \geq 1$. **Hint.** You will need to use the Hilbert Basis Theorem.

4. Find the Zariski closure of the following sets:
   (a) The projection of the hyperbola $V(xy - 1)$ in $\mathbb{R}^2$ onto the $x$-axis.
   (b) The boundary of the first quadrant in $\mathbb{R}^2$.
   (c) The set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

5. Let $I, J, K \subseteq \mathbb{k}[x_1, \ldots, x_n]$ be ideals. Prove the following:
   (a) $IJ \subseteq K$ if and only if $I \subseteq K : J$.
   (b) $(I : J) : K = I : JK$. 