1. (a) Let $\phi$ be the ring homomorphism $\mathbb{Z}[x, y] \to \mathbb{Z}[t]$ that sends $f(x, y) \mapsto f(t^6, t^8)$. Find, with proof, a generating set of minimum cardinality for the ideal $I = \ker \phi$ in $\mathbb{Z}[x, y]$.

(b) A prime ideal is minimal if it does not contain any smaller prime ideal. Find, with proof, all of the minimal prime ideals in the ring $\mathbb{Z}[x_1, x_2]/\langle x_1x_2 \rangle$.

(c) Show that the ideal in $\mathbb{Z}[x]$ generated by $x^2 + 1$ and 11 is maximal.

2. Prove or disprove the following statements about a commutative ring $R$ with 1, always interpreting “subring” to mean “subring with 1”:

(a) If $I$ is an ideal in $R$ and $R/I$ is a domain, then $R$ is also a domain.

(b) If $R$ is a UFD, then any subring of $R$ is a UFD.

(c) *If $R$ is a Euclidean domain, then any subring of $R$ is a Euclidean domain.

(d) *If $R$ is a UFD and $I$ is a prime ideal of $R$, then $R/I$ is a UFD.

(e) *If $R$ is a Noetherian ring, then any subring of $R$ is a Noetherian ring.

3. (a) *Prove $x^5 + y^7 + 1$ is irreducible in $\mathbb{Q}[x, y]$.

(b) *Prove $x^8 + y^3 + z^6$ is reducible in $\mathbb{F}_p[x, y, z]$ when $p = 2$, but irreducible in $\mathbb{F}_p[x, y, z]$ for all odd primes $p$.

4. Let $\mathbb{F}$ be a field. Then a polynomial $f(x) \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is homogeneous of degree $d$ if every monomial $x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ occurring in $f(x)$ with nonzero coefficient has the same degree $d = \sum_{i=1}^{n} a_i$. By segregating monomials according to their degree, one can express any polynomial $f$ uniquely as $f = f_0 + f_1 + \cdots + f_d$ with $f_i$ homogeneous of degree $i$. The $f_i$ are called the homogeneous components of $f$.

(a) Prove that for $f$ homogeneous of degree $d$ and $\lambda \in \mathbb{F}$, one has $f(\lambda x) = \lambda^d f(x)$.

(b) prove that for $f$ homogeneous of degree $d$, one has $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f(x)$.

(c) *An ideal $I$ in $\mathbb{F}[x_1, x_2, \ldots, x_n]$ is said to be homogeneous if every homogeneous component $f_i$ of any $f$ in $I$ also lies in $I$. Show that $I$ is a homogeneous ideal if and only if it can be generated by a collection of homogeneous polynomials.

(d) If $n = 2$ and $\mathbb{F}$ is algebraically closed (e.g. $\mathbb{F} = \mathbb{C}$), show that every homogeneous polynomial $f$ in $\mathbb{F}[x, y]$ in two variables can be factored as a product of linear (degree 1) polynomials, that is, $f(x, y) = \prod_{i=1}^{d}(\alpha_i x + \beta_i y)$ for some $\alpha_i, \beta_i$ in $\mathbb{F}$.