Boij–Söderberg theory and tensor complexes

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(joint work with Daniel Erman, Manoj Kummini, Steven V Sam)

The conjectures of M. Boij and J. Söderberg [3], proven by D. Eisenbud and F.-O. Schreyer [8] (see also [7, 4]), link the extremal properties of invariants of graded free resolutions of finitely generated modules over the polynomial ring $S = k[x_1, \dots, x_n]$ with the Herzog–Huneke–Srinivasan Multiplicity Conjectures. Here k is any field and S has the standard $\mathbb{Z}$-grading. In the course of their proof, Eisenbud and Schreyer introduce a groundbreaking relationship between the study of free resolutions over the S and the study of the cohomology of coherent sheaves on $\mathbb{P}_k^{n-1}$, via a nonnegative pairing of their associated numerics. This pairing has recently been categorified through work of Eisenbud and Erman [6], further extending the reach of Boij–Söderberg theory to larger classes of derived objects.

We now outline the main result of Boij–Söderberg theory for S. For simplicity, we restrict our attention to a graded S-module M that is of finite length; minor modifications yield the general situation. A minimal free resolution of M is an acyclic complex $(F_\bullet, \partial_\bullet)$ such that $H_0(F_\bullet) = M$, $\partial_i(F_i) \subseteq \langle x_1, \dots, x_n\rangle F_{i-1}$ for each i, and $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}}$. The ranks $\beta_{i,j}$ of the free modules are independent of the choice of resolution $F_\bullet$ of M, and they are called the *Betti numbers* of M. We record these Betti numbers into a *Betti table* for M, denoted $\beta(M)$.

It appears to be a difficult question to classify which integer tables can be realized as the Betti table of a graded S-module. In a shift in perspective, Boij and Söderberg suggested that this task be approached *up to scalar multiple.* In other words, view $\beta(M) \in \bigoplus_{i=0}^n \bigoplus_{j\in\mathbb{Z}} \mathbb{Q} =: \mathbb{V}$ and describe instead the *cone of Betti tables*

$$B_{\mathbb{Q}}(S) := \mathbb{Q}_{\geq 0} \cdot \{\, \beta(M) \mid M \text{ graded } S\text{-module of finite length} \,\}.$$

In this direction, we say that $d = (d_0 < d_1 < \cdots < d_n) \in \mathbb{Z}^{n+1}$ is a *degree sequence* for S, and we define a partial order on these sequences via $d \leq d'$ if $d_i \leq d'_i$ for all i. Associated to a degree sequence d, we define the *pure diagram* $\pi_d \in \mathbb{V}$ by

$$\beta_{i,j}(\pi_d) = \begin{cases} \frac{1}{\prod_{\ell\neq i} |d_i - d_\ell|} & \text{if } j = d_i, \\ 0 & \text{otherwise.} \end{cases}$$

These are related to the Herzog–Kühl equations for *pure free resolutions*, see [13].

Theorem 1 ([8]). *The extremal rays of $B_{\mathbb{Q}}(S)$ are precisely those spanned by the π_d. Furthermore, if $D \in B_{\mathbb{Q}}(S)$, then there exist $a_i \in \mathbb{Q}_{\geq 0}$ and degree sequences $d^1 < d^2 < \cdots < d^\ell$ such that $D = \sum_{i=1}^{\ell} a_i \pi_{d^i}$.*

The decomposition of D in Theorem 1, which endows $B_{\mathbb{Q}}(S)$ with the structure of a simplicial fan, arises from a greedy algorithm. An important ingredient in the proof of the theorem is to show that each pure diagram π_d is realizable, up to scalar multiple, as $\beta(M)$ for some module M. Originally, Eisenbud and Schreyer applied a nonconstructive pushforward argument to show the existence of such M.

A symmetrization of this argument produces *tensor complexes*, and the symmetry of these resolutions also allows them to be described explicitly.

Fix $(b_0, \ldots, b_n) \in \mathbb{N}^{n+1}$, let $R = \mathbb{Z}[x_{i,J}]$, where $1 \leq i \leq b_0$, $J = (j_1, \ldots, j_n)$, $1 \leq j_\ell \leq b_\ell$, and let $\phi = (x_{i,J}) \in R^{b_0} \otimes (R^{b_1})^* \otimes \cdots \otimes (R^{b_n})^*$ be the universal tensor. In [1], we construct, from the tensor ϕ and a choice $w \in \mathbb{Z}^{n+1}$ from an infinite family of appropriate weight vectors, a *tensor complex* $F(\phi, w)_\bullet$ with the following properties.

Theorem 2 ([1]). *A tensor complex $F(\phi, w)_\bullet$ satisfies the following:*

(i) *It is a graded pure free resolution of a Cohen–Macaulay module $M(\phi, w)$.*
(ii) *It is uniformly minimal over $\mathbb{Z}$, i.e., $F(\phi, w)_\bullet \otimes_R k[x_{i,J}]$ is a minimal free resolution for any field k.*
(iii) *It respects the multilinearity of ϕ, i.e., it is $\mathrm{GL}_{b_0} \times \cdots \times \mathrm{GL}_{b_n}$-equivariant.*
(iv) *Its differentials can by made explicit after fixing appropriate bases for the free modules $F(\phi, w)_i$.*

The construction of tensor complexes provides detailed new examples of minimal free resolutions, as well as a unifying view on a wide variety of complexes, including the Eagon–Northcott, Buchsbaum–Rim, and similar complexes, called Buchsbaum–Eisenbud matrix complexes [5, §A2.6], as well as the complexes used by Gelfand–Kapranov–Zelevinsky and Weyman to compute hyperdeterminants in [11, §14] and [14, §9.4]. In addition, tensor complexes have applications in Boij–Söderberg theory, as they provide infinitely many new families of pure resolutions, as well as the first explicit description of the differentials of the Eisenbud–Schreyer pure resolutions.

Corollary 3 ([1]). *There is a map $R \to S = k[x_1, \ldots, x_n]$ such that $S \otimes_R F(\phi, w)_\bullet$ is the Eisenbud–Schreyer pure resolution constructed in* [8]. *In particular, these resolutions can be made explicit.*

The `BoijSoederberg` and `TensorComplexes` packages of the computer algebra software `Macaulay2` contain implementations of the work discussed in this talk [12]. For surveys on Boij–Söderberg theory, see [2, 9, 10].

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