Problem 1. Let $E \subset \mathbb{R}$ have finite Lebesgue measure. Show that $m(E \cap [x, \infty)) \to 0$ as $x \to \infty$.

Proof. Define $f := \chi_E$. Then define $g(x) := \int_x^\infty f(t)dt = \int_x^\infty \chi_E(t)dt$. We see that $g(x) = m(E \cap [x, \infty))$.

Define $f_n := f \cdot \chi_{[-n,n]}$. So since $f_n \to f$ we can see that $|f_n| \to |f|$. $f_n$'s are measurable and increasing so therefore, by the Monotone Convergence Theorem we have that

$$\lim_{n \to \infty} \int \mathbb{R} |f_n| = \int \mathbb{R} \lim_{n \to \infty} |f_n| = \int \mathbb{R} |f|$$

Fix $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\int \mathbb{R} |f| - \int \mathbb{R} |f_n| < \varepsilon$$

Then for $g(x) = \int_x^\infty f(t)dt$, we have that $\forall n \geq N$,

$$g(n) = \int_n^\infty f(t)dt$$

$$\leq \int_n^\infty |f(t)|dt$$

$$\leq \int_{-\infty}^{-n} |f(t)|dt + \int_{n}^\infty |f(t)|dt$$

$$= \int_{-\infty}^\infty |f| - \int_{-\infty}^{n} |f(t)|dt$$

$$< \varepsilon$$

We can see that $g(x) = m(E \cap [x, \infty))$ and so since $g(x) \to 0$ we have that $m(E \cap [x, \infty)) \to 0$ as $x \to \infty$. \qed
**Problem 2.** Let $f : [0, 1] \to \mathbb{R}$ be a continuous function whose derivative exists almost everywhere and satisfies $f' \in L^1([0, 1])$. Prove or give a counterexample: $\int_0^1 f'(x)dx = f(1) - f(0)$.

False. Let $f$ be the Cantor-Lebesgue function (the Devil’s Staircase). By construction, $f : [0, 1] \to [0, 1] \subset \mathbb{R}$, and note that $f(0) = 0$ and $f(1) = 1$.

$f$ is the limit of a sequence of increasing, continuous functions $\{f_n\}_{n \in \mathbb{N}}$ such that

$$|f_{n+1} - f_n| < 2^{-n-1}.$$ 

Thus $f_n \to f$ uniformly and so $f$ is continuous.

Let $\mathcal{C}$ denote the standard middle-thirds Cantor set. We know that $m(\mathcal{C}) = 0$. By construction, $f'(x) = 0 \ \forall x \in [0, 1] \setminus \mathcal{C}$. So we see that $f'$ exists almost everywhere (on all but a set of measure 0). $0 \in L^1([0, 1]) \implies f' \in L^1([0, 1])$.

Thus $f$ satisfies all properties given, however

$$\int_0^1 f'(x)dx = \int_0^1 0dx = 0$$

$$f(1) - f(0) = 1 - 0 = 1$$

and since $0 \neq 1$, we have that the Cantor-Lebesgue function is a counterexample.
Problem 3. Let $E$ be a closed Lebesgue measurable subset of $[0, 1]$. Prove or disprove:

a. If $E$ has measure 0, then $E$ is nowhere dense.

b. If $E$ is nowhere dense, then $E$ has measure 0.

---

a. True.

*Proof.* Since $E$ is closed, $\text{int}(\overline{E}) = \text{int}(E)$. Since $E$ is measure 0, $E$ cannot contain any intervals. So, $\text{int}(E) = \emptyset$. Thus, $E$ is nowhere dense. \qed

b. False.

The Fat-Cantor Set - $C_s$. First note that the $C_s$ has measure $1/2$. Now we will show it is nowhere dense. The complement of the $C_s$ is open, therefore the $C_s$ is closed. This implies that $\text{int}(\overline{C_s}) = \text{int}(C_s)$. But, $C_s$ contains no intervals and thus the interior of $C_s$ is the empty set $\implies C_s$ is nowhere dense.
Problem 4. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be absolutely continuous, and assume that \( f' \in L^2([0, 1]) \) and that \( f(0) = 0 \). Show that the following limit exists, and compute its value.

\[
\lim_{x \to 0^+} x^{-1/2} f(x)
\]

Proof. Since \( f \) is absolutely continuous, we have that it satisfies the Second Fundamental Theorem of Calculus. Thus we can say that

\[
f(x) = \int_0^x f'(t)dt \quad \Rightarrow \quad f(x) = \int_0^x f'(t)dt
\]

Note that \( f' \in L^2([0, 1]) \) \( \Rightarrow \left( \int_0^1 |f'(t)|^2dt \right)^{1/2} < \infty \).

\[
|f(x)| = \left| \int_0^x f'(t)dt \right| \\
\leq \int_0^x |f'(t)| \cdot |1|dt \\
\leq \left( \int_0^x |f'(t)|^2dt \right)^{1/2} \left( \int_0^x |1|^2dt \right)^{1/2} \quad \text{by Hölder’s Inequality} \\
= \left( \int_0^x |f'(t)|^2dt \right)^{1/2} \cdot x^{1/2}
\]

\[
\Rightarrow |x^{-1/2} f(x)| \leq \left( \int_0^x |f'(t)|^2dt \right)^{1/2}
\]

Now we want to take the limit as \( x \) goes to 0 from the right. Note that since we are dealing with absolute values, we have that all our quantities are greater than 0.

\[
0 \leq \lim_{x \to 0^+} |x^{-1/2} f(x)| \leq \lim_{x \to 0^+} \left( \int_0^x |f'(t)|^2dt \right)^{1/2}
\]

We claim that \( \left( \int_0^x |f'(t)|^2dt \right)^{1/2} \rightarrow 0 \) as \( x \rightarrow 0^+ \). It suffices to show that \( \left( \int_0^x |f'(t)|^2dt \right) \rightarrow 0 \) as \( x \rightarrow 0^+ \). Note that \( |f'(t)| \cdot \chi_{[0,x]} \leq |f'(t)| \in L^2([0, 1]) \). So we see by Lebesgue Dominated Convergence,

\[
\lim_{x \to 0^+} \left( \int_0^x |f'(t)|^2dt \right) = \lim_{x \to 0^+} \left( \int_0^\infty |f'(t)|^2 \chi_{[0,x]}dt \right) = \int_0^\infty \lim_{x \to 0^+} |f'(t)|^2 \cdot \chi_{[0,x]}dt \int_0^\infty 0dt = 0
\]

Therefore, by the Squeeze Theorem, we have that the limit exists and

\[
\lim_{x \to 0^+} |x^{-1/2} f(x)| = 0 \quad \Rightarrow \quad \lim_{x \to 0^+} x^{-1/2} f(x) = 0
\]
Problem 5. Let $K$ be a compact subset of $\mathbb{R}$, and suppose that $f : K \to \mathbb{R}$ and $f_n : K \to \mathbb{R}$, $n = 1, 2, \ldots$, are continuous. Suppose that, for every $x \in K$, $f_{n+1}(x) \leq f_n(x)$, $n = 1, 2, \ldots$, and $\lim_{n \to \infty} f_n(x) = f(x)$.

a. Show that $f_n(x) \to f(x)$ uniformly on $K$ as $n \to \infty$.

b. Give an example to show the compactness of $K$ is necessary.

a. Proof. Define the sequence $F_n(x) := |f_n(x) - f(x)|$. This is continuous as it is the composition of a continuous function, $|x|$, with the difference of two continuous functions.

We observe that

$$\max_{x \in K} \{F_n(x)\} = \sup_{x \in K} |f_n(x) - f(x)|.$$

We can easily see then that $\max_{x \in K} \{F_n(x)\}$ is bounded below by 0. Since $\{f_n\}$ is decreasing $\implies \{F_n\}$ is also decreasing. Since it is bounded below and decreasing, it has a limit.

Claim: $F_n \to 0$, uniformly.

Suppose for contradiction that $F_n$ does not converge uniformly to 0. There there exists some $\varepsilon_0$ and a sequence $\{x_n\}$ such that for all $n \in \mathbb{N}$,

$$|f_n(x_n) - f(x_n)| \geq \varepsilon_0.$$

Since $\{x_n\}$ is defined on a compact set $K$, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in K$ (since $K$ is closed by the Heine-Borel Theorem). Thus, there exists an $n \in \mathbb{N}$ such that $\forall n \geq N$,

$$F_n(x_0) = |f_n(x_0) - f(x_0)| \geq \varepsilon_0$$

which is a contradiction since $f_n(x) \to f(x)$ pointwise. Thus, $F_n \to 0$ uniformly which implies that $f_n(x) \to f(x)$ uniformly on $K$ as $n \to \infty$.

b. Let $X = (0, 1)$. Clearly $X$ is not compact. Let $f_n : [0, 1] \to \mathbb{R}$ be the sequence of functions where $f_n(1) = f_n(0) = 1$, and $f_n$ is 0 on the middle $\frac{n}{n+1}$’s of the interval and $f_n$ is continuous.

$f_n|_X$ is also continuous and $f_n \to 0$.

However, $\sup_{x \in X} |f_n(x) - 0| = 1$, so the $f_n$s do not converge uniformly.
Problem 6. Using an appropriate Fourier series, show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6} \).

First we note that by definition we have:

\[
\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}
\]

And also note that the Fourier coefficients, \(a_n\) and Fourier series, \(f(x)\), is defined as follows:

\[
a_n = \frac{1}{2k} \int_{-k}^{k} f(x)e^{-i\pi nx} \, dx, \quad f(x) = \sum_{n=-\infty}^{\infty} a_n e^{i\pi nx}
\]

where \(2k\) is the period.

Parseval’s Identity can be stated as follows:

\[
\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2k} \int_{-k}^{k} |f(x)|^2 \, dx
\]

Now define \(f(x) := x\) on \([-\pi, \pi]\). We now want to find the Fourier coefficients. Note that \(e^{-inx}\) is \(2\pi\) periodic because it is \(\cos(nx)\).

\[
\int_{-\pi}^{\pi} x e^{-inx} \, dx = \frac{1}{2\pi} \left( -\frac{x}{in} e^{-inx} \bigg|_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-inx} \, dx \right)
\]

\[
= -\frac{1}{2\pi in} \left( xe^{-in\pi} \bigg|_{-\pi}^{\pi} \right)
\]

\[
= -\frac{1}{2\pi in} (e^{-in\pi} + e^{in\pi})
\]

\[
= -\frac{\cos(n\pi)}{in} = \frac{(-1)^{n+1}}{in}
\]

Thus we see by Parseval’s Identity that

\[
\sum_{n=-\infty}^{\infty} \left| \frac{(-1)^{n+1}}{in} \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|^2 \, dx
\]

\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2} = \frac{1}{2\pi} \left( \frac{\pi^3}{3} \right)
\]

\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2} = \frac{1}{6\pi} (\pi^3 + \pi^3) = \frac{\pi^2}{3}
\]

And so we have the following

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}
\]
Problem 7. Assume that $E$ is a bounded Lebesgue measurable subset of $\mathbb{R}$, and let
\[ f(t) = \int_E \cos(tx)dx, \quad t \in \mathbb{R} \]
Prove or disprove:

a. $f$ has compact support
b. $f$ is continuously differentiable

a. False.

Let $E := \left[ 0, \frac{\pi}{2} \right]$. Then we have that
\[ f(t) = \int_0^{\frac{\pi}{2}} \cos(tx)dx = \frac{1}{t} \sin(tx) \bigg|_0^{\frac{\pi}{2}} = \frac{1}{t} \sin \left( \frac{t}{2} \pi \right) \]
So we note that if $t$ is an even number, then $f(t) = 0$. Let us denote the set of even numbers by $2\mathbb{Z}$. Thus we see that the support of $f$ if the set $\mathbb{R} \setminus 2\mathbb{Z}$, which is not a compact set.

b. True.

Proof.
\[
\frac{d}{dt} f(t) = \frac{d}{dt} \int_E \cos(tx)dx
= \lim_{h \to 0} \frac{\int_E \cos((t + h)x) - \int_E \cos(tx)}{h}
= \lim_{h \to 0} \int_E \frac{\cos((t + h)x) - \cos(tx)}{h} dx
\]
We want to be able to pass the limit in so we will set up to use the Lebesgue Dominated Convergence Theorem. Let $\{h_n\}$ be a sequence such that $h_n \to 0$ as $n \to \infty$. Fix $t \in \mathbb{R}$, $n \in \mathbb{N}$, and $x \in E$. Thus by the Mean Value Theorem we have that
\[
\left| \frac{\cos((t + h_n)x) - \cos(tx)}{h_n} \right| = |x \sin(\xi x)| \leq |x| \leq M
\]
where $\xi \in [t, t + h_n]$ and $M := \sup_{x \in E} |x|$, which exists because $E$ is bounded. By the Lebesgue Dominated Convergence Theorem,
\[
\frac{d}{dt} f(t) = \lim_{h \to 0} \int_E \frac{\cos((t + h)x) - \cos(tx)}{h} dx = \int_E \lim_{h \to 0} \frac{\cos((t + h)x) - \cos(tx)}{h} dx
\]
\[ = \int_E -x \sin(tx)dx < \infty \]
since $E$ is bounded. So we have that $-x \sin(tx)$ is integrable and thus we have found the derivative of $f(t)$.
Now we want to show that $f'(t)$ is continuous. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence such that $t_n \to t_0$ as $n \to \infty$. We want that
\[
\lim_{n \to \infty} f'(t_n) = f'(t_0).
\]
We know that $| -x \sin(tx)| \leq M$ and so we can once again pass the limit into the integral by the Lebesgue Dominated Convergence Theorem. Note also that $\sin(x)$ is a continuous function so we can pass the limit into $\sin(x)$ as well. Thus we get the following:
\[
\lim_{n \to \infty} f'(t_n) = \lim_{n \to \infty} \int_E -x \sin(t_n x) \, dx = \int_E \lim_{n \to \infty} -x \sin(t_n x) \, dx = \int_E -x \sin(t_0 x) \, dx = f'(t_0).
\]
So $f'(t)$ is continuous.
Therefore, $f(t)$ is continuously differentiable. 
\[\square\]