Problem 1. Let $E$ and $F$ be Lebesgue measurable subsets of $\mathbb{R}$ with $E \subset F$. Suppose that $F$ is Borel measurable, but suppose that $E$ is not Borel measurable. Prove or disprove: $F \cap E^c$ is uncountable.

True.

Proof. Assume for contradiction that $F \cap E^c$ is countable. Then $F \cap E^c$ is Borel measurable as it must be made up of the countable union of points, which are closed. Then we can say $E^c = (F \cap E^c) \cup F^c$. Since the Borel $\sigma$-algebra is closed under complements, $F^c$ is Borel measurable. Since the Borel $\sigma$-algebra is closed under countable unions, we have that $E^c$ is Borel measurable. Which would imply that $E$ is Borel measurable since the Borel $\sigma$-algebra is closed under complements - which is a contradiction. \qed
Problem 2. a. Construct an example of a set $U \subset [0, 1]$ such that $U$ is open and dense and such that $m(U) < 1$, where $m$ denotes Lebesgue measure.

b. Let $E$ be a countable dense subset of $[0, 1]$, and let $U$ be an open dense subset of $[0, 1]$ satisfying $m(U) < 1$. Prove or disprove: $E \cap U^c \neq \emptyset$.

a. Let $C_*$ denote the Fat-Cantor Set (Smith-Volterra Cantor Set). We know that $m(C_*) = \frac{1}{2}$. So let $U := [0, 1] \setminus C_*$. $C_*$ is closed and contains no intervals, therefore it is nowhere dense. This implies that $U$ is open and dense. $m(U) = m([0, 1]) - m(C_*) = 1 - \frac{1}{2} = \frac{1}{2}$.

b. False.

Define $E := \mathbb{Q}$. Now let $\{q_n\}_{n=1}^\infty$ be an enumeration of the rationals in $[0, 1]$. Cover $q_n$ by an interval of radius $2^{-n-1} =: I_n$. Then let $U := \bigcup_{n=1}^\infty I_n$. Then we see that $m(U) < 1$. It is clear that $U^c$ does not contain any rationals. Therefore, $E \cap U^c = \emptyset$. 
Problem 3. a. State Parseval’s Identity for Fourier series of functions on \([0, 1]\).

b. Using Parseval’s Identity, show that \(\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\).

a. Let \(f \in L^2([0, 1])\). Then, let \(a_n\) be
\[
a_n = \int_0^1 f(x)e^{-\pi i nx} \, dx.
\]

Parseval’s Identity states that
\[
\sum_{n \in \mathbb{Z}} |a_n|^2 = \int_0^1 |f(x)|^2 \, dx.
\]

b. The Fourier coefficients, \(a_n\) and Fourier series for \(f(x)\), is defined as follows:
\[
a_n = \frac{1}{2k} \int_{-k}^{k} f(x)e^{-\frac{\pi i nx}{2k}} \, dx, \quad f(x) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{\pi i nx}{2k}}
\]
where \(2k\) is the period.

Now define \(f(x) := x\) on \([-\pi, \pi]\). We now want to find the Fourier coefficients. Note that \(e^{-inx}\) is \(2\pi\) periodic because it is \(\cos(nx)\).
\[
a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{-inx} \, dx = \frac{1}{2\pi} \left( \frac{-x}{in} e^{-inx} \bigg|_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{-inx} \, dx \right)
\]
\[
= -\frac{1}{2\pi in} \left( xe^{-inx} \bigg|_{-\pi}^{\pi} \right)
\]
\[
= -\frac{1}{2in} (e^{-in\pi} + e^{in\pi})
\]
\[
= -\frac{\cos(n\pi)}{in} = \frac{(-1)^{n+1}}{in}
\]

Thus we see by Parseval’s Identity that
\[
\sum_{n=-\infty}^{\infty} \left| \frac{(-1)^{n+1}}{in} \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|^2 \, dx
\]
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2} = \frac{1}{2\pi} \left( \frac{1}{3} \right|_{-\pi}^{\pi} = \frac{1}{2\pi} \left( \pi^3 + \pi^3 \right) = \frac{\pi^2}{3}
\]

And so we have the following
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}
\]
**Problem 4.**

a. State the Riemann-Lebesgue Lemma for Fourier transforms of functions on \( \mathbb{R} \).

b. Let \( E \subset \mathbb{R} \) have finite Lebesgue measure. Show that \( f(t) = \int_E \sin(tx)dx \) is a continuous function.

c. How differentiable is \( f \) if \( E \) is also bounded?

da. Let \( f \in L^1(\mathbb{R}) \). Then the Riemann-Lebesgue Lemma states that

\[
\int_{\mathbb{R}} f(x)e^{-i\xi x}dx \to 0 \quad \text{as} \quad |\xi| \to \infty.
\]

b. *Proof.* Let \( \{t_n\} \) be a sequence such that \( t_n \to t_0 \) as \( n \to \infty \). We want to show that

\[
\lim_{n \to \infty} f(t_n) = f(t_0).
\]

We see that \( |\sin(t_nx)| \leq 1 \in L^1(E) \) since \( E \) has finite measure. We also note that the sine function is continuous. So we observe

\[
\lim_{n \to \infty} f(t_n) = \lim_{n \to \infty} \int_E \sin(t_n x)dx = \int_E \lim_{n \to \infty} \sin(t_n x)dx = \int_E \sin(t_0 x)dx = f(t_0)
\]

c. Infinitely?
Problem 5. Suppose that \( F : [a, b] \to \mathbb{R} \) and \( G : [a, b] \to \mathbb{R} \) are absolutely continuous. Show that \( FG \) is absolutely continuous.

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Proof. Fix \( \varepsilon > 0 \). Since \( F \) and \( G \) are on a bounded interval, so we define:

\[
\varepsilon_1 := \frac{\varepsilon}{\max(G)} \quad \varepsilon_2 := \frac{\varepsilon}{\max(F)}
\]

Since \( F \) and \( G \) are absolutely continuous, \( \exists \delta_1, \delta_2 \) such that

\[
\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \varepsilon_1 \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta_1
\]

\[
\sum_{k=1}^{N} |G(b_k) - G(a_k)| < \varepsilon_1 \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta_1
\]

So we want to show that

\[
\sum_{k=1}^{N} |FG(b_k) - FG(a_k)| < 2\varepsilon \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \min \{\delta_1, \delta_2\}
\]

\[
= \sum_{k=1}^{N} |F(a_k)G(a_k) - F(b_k)G(b_k) + F(a_k)G(b_k) - F(a_k)G(b_k)|
\]

\[
= \sum_{k=1}^{N} |F(a_k)(G(a_k) - G(b_k)) + G(b_k)(F(a_k) - F(b_k))|
\]

\[
\leq \sum_{k=1}^{N} \max(F)|G(a_k) - G(b_k)| + \sum_{k=1}^{\infty} \max(G)|F(a_k) - F(b_k)|
\]

\[
\leq \max(F) \cdot \varepsilon_2 + \max(F) \cdot \varepsilon_1 < 2\varepsilon
\]
Problem 6. Let \( f_n \) be a sequence of nonnegative functions in \( L^3((0,1)) \) such that \( \|f_n\|_3 = 1 \), for all \( n \), and \( f_n \to 0 \) a.e. as \( n \to \infty \). Prove that

\[
\int f_n \to 0 \text{ as } n \to \infty.
\]

Proof. Let \( \Omega := (0,1) \). Since \( \mu(\Omega) = 1 < \infty \), we have that \( f_n \to 0 \) a.e. implies \( f_n \to 0 \) in measure. That is to say that \( \forall \varepsilon > 0, \mu(\{|f_n| \geq \varepsilon\}) \to 0 \) as \( n \to \infty \).

So fix \( \varepsilon > 0 \). Then \( \exists N \in \mathbb{N} \) such that \( \forall n \geq N \), we have that

\[
\mu(\{|f_n| \geq \varepsilon\}) < \varepsilon^{\frac{3}{2}}.
\]

Now we will use this fact along with Hölder’s Inequality for \( p = 3 \) and \( q = \frac{3}{2} \), so \( \forall n \geq N \), we have that

\[
\left|\int_0^1 f_n d\mu\right| \leq \int_0^1 |f_n| d\mu
\]

\[
= \int_{\{|f_n| < \varepsilon\}} |f_n| d\mu + \int_{\{|f_n| \geq \varepsilon\}} |f_n| d\mu
\]

\[
< \int_{\{|f_n| < \varepsilon\}} \varepsilon d\mu + \int_0^1 |f_n| \cdot \chi_{\{|f_n| \geq \varepsilon\}} d\mu
\]

\[
< \varepsilon \cdot \mu(\Omega) + \left(\int_0^1 |f_n|^3 d\mu\right)^{\frac{1}{3}} \left(\int_0^1 \left(\chi_{\{|f_n| \geq \varepsilon\}}\right)^{\frac{3}{2}} d\mu\right)^{\frac{2}{3}}
\]

\[
\leq \varepsilon + \mu(\{|f_n| \geq \varepsilon\})^{\frac{2}{3}}
\]

\[
< \varepsilon + \varepsilon = 2\varepsilon
\]

Thus we see that \( \int f_n \to 0 \) as \( n \to \infty \). \( \square \)
Problem 7. Let \( f, g \in L^1(\mathbb{R}) \), and let \( h = f \ast g \). Suppose that \( \int yf(y)dy = 0 \) and that \( \int yg(y)dy = 0 \). Prove that \( \int yh(y)dy = 0 \).

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Proof. Since \( f, g \in L^1(\mathbb{R}) \), we know that \( f \) and \( g \) are measurable. This also tells us that \(|f(y - t)|\) and \(|g(t)|\) are measurable. We also note that \(|y|\) is measurable. Thus we see that \(|yf(y - t)g(t)| \geq 0\) is measurable as it is the product and composition of measurable functions. Thus by Tonelli’s Theorem, we have that

\[
\int_{\mathbb{R}^2} |yf(y - t)g(t)|d(x, y) < \infty.
\]

Thus, this tells us that \( yf(y - t)g(t) \) is integrable on \( \mathbb{R}^2 \). So by Fubini’s Theorem we know that

\[
\int \int_{\mathbb{R}} yf(y - t)g(t)dydt = \int \int_{\mathbb{R}} yf(y - t)g(t)dydt = \int_{\mathbb{R}^2} yf(y - t)g(t)d(y,t)
\]

We want to look at

\[
\int_{\mathbb{R}} yh(y)dy = \int_{\mathbb{R}} y(f \ast g)(y)dy = \int_{\mathbb{R}} y \left( \int_{\mathbb{R}} f(y - t)g(t)dt \right) dy
\]

Consider,

\[
\int \int_{\mathbb{R}} yf(y - t)g(t)dtdy = \int \int_{\mathbb{R}} yf(y - t)g(t)dydt
\]

\[
= \int \left( \int yf(y - t)dy \right) g(t)dt
\]

let \( x = y - t \quad \Rightarrow \quad dx = dy \)

\[
= \int \left( \int (x + t)f(x)dx \right) g(t)dt
\]

\[
= \int \left( \int xf(x) + tf(x)dx \right) g(t)dt
\]

\[
= \int \left( \int xf(x)dx + \int tf(x)dx \right) g(t)dt
\]

\[
= \int \left( \int tf(x)dx \right) g(t)dt
\]

\[
= \int \left( \int tg(t)dt \right) f(x)dx
\]

\[
= 0
\]

\( \square \)