Problem 1. Let $m$ denote the Lebesgue measure on $\mathbb{R}$, and let $E \subset \mathbb{R}$ be Lebesgue measurable with $m(E) < \infty$. Show that $m(E \cap [x, \infty)) \to 0$ as $x \to \infty$.

Proof. Define $f := \chi_E$. Then define $g(x) := \int_x^\infty f(t)dt$. Notice that $g(x) = m(E \cap [x, \infty))$.

Define $f_n := \chi_{[-n, n]}$. We see that $f_n \to f$ as $n \to \infty$. This tells us then that $|f_n| \to |f|$. Since the $f_n$’s are measurable and increasing, we have by the Monotone Convergence Theorem that

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f_n| = \int_{\mathbb{R}} \lim_{n \to \infty} |f_n| = \int_{\mathbb{R}} |f|$$

And so we have that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\int_{\mathbb{R}} |f| - \int_{\mathbb{R}} |f_n| < \varepsilon.$$ 

And so we have $\forall n \geq N$,

$$g(n) = \int_{n}^{\infty} f(t)dt$$
$$\leq \int_{n}^{\infty} |f(t)|dt$$
$$\leq \int_{-\infty}^{-n} |f(t)|dt + \int_{n}^{\infty} |f(t)|dt$$
$$= \int_{\mathbb{R}} |f(t)|dt - \int_{\mathbb{R}} |f_n(t)|dt$$
$$< \varepsilon$$

and so we have that $g(x) \to 0$ as $x \to \infty$ which implies that $m(E \cap [x, \infty)) \to 0$ as $x \to \infty$. \qed
Problem 2. Assuming that $f \in L^2([0,1])$ and that $f(x) \geq 0$ for $x \in [0,1]$, show that the following integral is finite.

\[
\int_0^1 \sqrt[3]{\frac{f(x)}{x}} \, dx
\]

Proof. We will use Hölder’s inequality with $p = 4$ and $q = \frac{4}{3}$.

\[
\int_0^1 \sqrt[3]{f(x)} \cdot x^{-\frac{1}{2}} dx \leq \left( \int_0^1 |\sqrt[3]{f(x)}|^4 dx \right)^{\frac{1}{4}} \left( \int_0^1 |x^{-\frac{1}{2}}|^\frac{4}{3} dx \right)^{\frac{3}{4}}
\]

\[
= \left( \int_0^1 |\sqrt[3]{f(x)}|^4 dx \right)^{\frac{1}{4}} \left( \int_0^1 x^{- \frac{2}{3}} dx \right)^{\frac{3}{4}}
\]

\[
= \left( \left( \int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}} \right)^{\frac{3}{2}} \left( 3x^{\frac{1}{3}} \right)^{\frac{1}{4}}
\]

\[
= \|f\|_{L^2([0,1])} \cdot 3^{\frac{3}{4}} < \infty
\]

Thus we see that \( \int_0^1 \sqrt[3]{\frac{f(x)}{x}} \, dx < \infty \). $\square$
Problem 3. Prove or disprove:

a. Every compact metric space is complete.

b. Every complete metric space is locally compact.

d. True.

This follows from the Heine-Borel Theorem which states that a metric space is compact iff it is complete and totally bounded. Therefore every compact metric space is complete.

b. False.

Let $H$ be an infinite dimensional Hilbert space. We know $H$ is a complete metric space. Assume for contradiction that it is locally compact. Then we know that there exists a compact neighborhood $K$ of the origin, $O$. Then $\exists B_r(O) \subset K$, compact. Choose $\{e_n\}_{n=1}^{\infty}$, an orthonormal sequence with norm 1. So then we see that $\{re_n\}_{n=1}^{\infty} \subset B_r(O)$. By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence $\{re_{n_k}\} \subset B_r(O)$. In particular, the sequence is Cauchy. Thus,

$$\|re_{n_k} - re_{n_\ell}\|^2 = \|re_{n_k}\|^2 + 2\langle re_{n_k}, re_{n_\ell} \rangle + \|re_{n_\ell}\|^2 = 2r^2$$

$$\implies \|re_{n_k} - re_{n_\ell}\| = r\sqrt{2}$$

and is therefore not Cauchy. This is a contradiction and thus $H$ cannot be locally compact.
Problem 4. Let $K$ be a compact subset of $\mathbb{R}$, and suppose that $f : K \to \mathbb{R}$ and $f_n : K \to \mathbb{R}$, $n = 1, 2, \ldots$, are continuous. Suppose that, for every $x \in K$, $f_{n+1}(x) \leq f_n(x)$, $n = 1, 2, \ldots$, and $\lim_{n \to \infty} f_n(x) = f(x)$.

a. Show that $f_n(x) \to f(x)$ uniformly on $K$ as $n \to \infty$.

b. Give an example to show that the compactness of $K$ is necessary.

a. Proof. Fix $\varepsilon > 0$. Let $g_n(x) = f_n(x) - f(x)$. Then $g_n(x)$ is positive because $f_n(x) \geq f(x)$ for every $x$ and $g_n(x)$ is continuous for every $n$ since $f$ and $f_n$, $n = 1, 2, \ldots$, are all continuous. Let $E_m = \{x : |g_m(x)| < \varepsilon\}$. Note that $E_m$ is open because it is the preimage of $(-\varepsilon, \varepsilon)$ under $g_m$, which is continuous. Also notice that because $f_n(x)$ converges pointwise to $f(x)$, every $x$ is in $E_m$ for some $m \in \mathbb{N}$. Thus $K = \bigcup_{m=1}^{\infty} E_m$. $K$ is compact, so $\exists$ a finite cover $E_{i_1}, \ldots, E_{i_N}$. Since the $g_m$s decrease, $E_1 \subseteq E_2 \subseteq \ldots$. This means for any $x \in K$, $x \in E_{m_N}$ so $|g_{m_N}(x)| = |f_{m_N}(x) - f(x)| < \varepsilon$. Thus $f_n \to f$ uniformly on $K$ as $n \to \infty$.

b. Consider the sequence of functions $f_n(x) = x^n$ on the open interval $(0, 1)$. Then $f_n(x) \to f(x) = 0$ as $n \to \infty$ and $f$, and $f_n$, $n = 1, 2, \ldots$ are all continuous on $(0, 1)$. However, $f_n(x)$ does not converge uniformly to 0 because for any $n$, there is an $x \in (0, 1)$ such that $x^n > \frac{1}{2}$. Thus the compactness of $K$ is necessary.
Problem 5. Consider a function $f : \mathbb{R} \to \mathbb{R}$ which is periodic with period one and which satisfies

$$f(x) = \chi_{(0,1/2)}(x) - \chi_{(-1/2,0)}(x) \quad \text{for} \quad |x| \leq \frac{1}{2}$$

a. Compute the Fourier series for $f$.

b. Use your result from part (a) to compute $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

a. First we calculate the Fourier exponents, $a_n$:

$$a_n = - \int_{-1/2}^{0} e^{-2\pi inx} dx + \int_{0}^{1/2} e^{-2\pi inx} dx$$

$$= \left. \frac{1}{2\pi in} e^{-2\pi inx} \right|_{-1/2}^{0} + \left. \frac{1}{2\pi in} e^{-2\pi inx} \right|_{0}^{1/2}$$

$$= \left( \frac{1}{2\pi in} - \frac{1}{2\pi in} e^{-\pi in} \right) + \left( -\frac{1}{2\pi in} e^{-\pi in} + \frac{1}{2\pi in} \right)$$

$$= \frac{1}{\pi in} - \frac{1}{2\pi in} (e^{\pi in} + e^{-\pi in})$$

$$= \frac{1}{\pi in} - \frac{1}{2\pi in} (2\cos(\pi n))$$

$$= \frac{1 - (-1)^n}{\pi in}$$

And so the Fourier series of $f(x)$ is

$$\sum_{n \in \mathbb{Z}, n \neq 0} \frac{1 - (-1)^n}{\pi in} e^{\pi inx}$$

b. Note that if $n$ is even, then we get $1 - (-1)^n = 0$ and if $n$ is odd then $1 - (-1)^n = 2$. Thus we can rewrite the Fourier series of $f$ to only account for the odd integers as follows:

$$\sum_{n \in \mathbb{Z}, n \neq 0} \frac{2}{\pi i(2n-1)} e^{2\pi i(2n-1)}$$

Now by Parseval’s Identity we have that

$$\sum_{n=-\infty}^{\infty} \left| \frac{2}{\pi i(2n-1)} \right|^2 = \int_{-1/2}^{1/2} |f(x)|^2 dx$$

So we see that

$$\sum_{n=-\infty}^{\infty} \left| \frac{2}{\pi i(2n-1)} \right|^2 = \frac{4}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2}$$
\[ \int_{-1/2}^{1/2} |f(x)|^2 \, dx = 1 \]

So thus we have that
\[
\frac{4}{\pi^2} \sum_{n \to \infty}^{\infty} \frac{1}{(2n-1)^2} = 1
\]
\[
\sum_{n \to \infty}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{4}
\]
\[
\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{2} \sum_{n \to \infty}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{2} \cdot \frac{\pi^2}{4} = \frac{\pi^2}{8}
\]
Problem 6. Consider the function \( g : \mathbb{R} \to \mathbb{R} \) given by

\[
g(x) = \int_{-1}^{1} \frac{(1-t^2)^2 \cos(xt)}{1+t^2} \, dt.
\]

Prove or disprove:

a. \( x^2 g(x) \to 0 \) as \( x \to \infty \).

b. \( g \in C^2(\mathbb{R}) \).

---

a. True.

Proof. Define

\[
f(t) := \frac{(1 - t^2)^2}{1 + t^2}.
\]

We see then that

\[
f'(t) = \frac{2t(t^2 - 1)(t^2 + 3)}{(t^2 + 1)^2}.
\]

Now note that \( f'(1) = f'(-1) = 0 \) and also \( f(1) = f(-1) = 0 \).

Consider the function \( f''(t) \cdot \chi_{[-1,1]} \). Clearly this function is in \( L^1(\mathbb{R}) \). Thus, by the Riemann-Lebesgue Lemma we have that

\[
\lim_{x \to \infty} \int_{\mathbb{R}} f''(t) \cdot \chi_{[-1,1]} \cdot \cos(tx) \, dt = 0
\]

Now, we integrate by parts twice and get,

\[
\int_{\mathbb{R}} f''(t) \cdot \chi_{[-1,1]} \cdot \cos(tx) \, dt = \int_{-1}^{1} f''(t) \cos(tx) \, dt
\]

\[
= f'(t) \cos(tx) \bigg|_{-1}^{1} + x \int_{-1}^{1} f'(t) \sin(tx) \, dt
\]

\[
= x \int_{-1}^{1} f'(t) \sin(tx) \, dt
\]

\[
= x \left( f(t) \sin(tx) \bigg|_{-1}^{1} - x \int_{-1}^{1} f(t) \cos(tx) \, dt \right)
\]

\[
= -x^2 \int_{-1}^{1} f(t) \cos(tx) \, dx
\]

\[
= -x^2 g(x)
\]
and so we have
\[
\lim_{x \to \infty} \int_{\mathbb{R}} f''(t) \cdot \chi_{[-1,1]} \cdot \cos(tx) \, dt = \lim_{x \to \infty} -x^2 g(x) = 0
\]

\[\implies \lim_{x \to \infty} x^2 g(x) = 0.\]

\[\square\]

b. True.

**Proof.** We need the second derivative of \( g \) to exist and be continuous. We will first show existence. Define \( f(t) \) as in part (a). Then, we have that
\[
\frac{d}{dx}g = \lim_{h \to 0} \frac{\int_{-1}^{1} f(t \cos(tx)) \, dt - \int_{-1}^{1} f(t) \, dt}{h}
\]
\[
= \lim_{h \to 0} \int_{-1}^{1} \frac{\cos(t + h) \cos(tx) - \cos(tx)}{h} f(t) \, dt
\]
Let \( \{h_n\} \) be a sequence such that \( h_n \to 0 \) as \( n \to \infty \). Fix \( t, x \in \mathbb{R} \) and \( n \in \mathbb{N} \). We know that the function \( \cos(x) \) is continuous on the interval \([x, x + h_n]\) and therefore by the Mean Value Theorem, there exists a \( \xi \in (x, x + h_n) \) such that
\[
\left| \frac{\cos(t(x + h_n)) - \cos(tx)}{h_n} \right| = |t \sin(t \xi)| \leq |t| \leq 1
\]
since \( t \in [-1, 1] \). Thus we have that
\[
\left| \frac{\cos(t + h) \cos(tx) - \cos(tx)}{h} f(t) \right| \leq |f(t)|
\]
where we know that \( f(t) \in L^1(\mathbb{R}) \). Therefore, by the Lebesgue Dominated Convergence Theorem we can pass the limit into the integral,
\[
\frac{d}{dx}g = \lim_{h \to 0} \int_{-1}^{1} \frac{\cos(t(x + h)) - \cos(tx)}{h} f(t) \, dt = \int_{-1}^{1} \lim_{h \to 0} \frac{\cos(t(x + h)) - \cos(tx)}{h} f(t) \, dt
\]
\[
= \int_{-1}^{1} -t \sin(tx) f(t) \, dt < \infty
\]
since \(-t \sin(tx) f(t) \in L^1([-1, 1])\) and thus we have shown that \( g' \) exists. By a very similar argument one can show that \( g'' \) exists and it can be shown that
\[
g''(x) = \int_{-1}^{1} -t^2 \cos(tx) f(t)
\]
Now we want to show that $g''(x)$ is continuous $\forall x \in \mathbb{R}$. Let $\{x_n\}$ be a sequence such that $x_n \to x_0$ as $n \to \infty$. We want to show that

$$\lim_{n \to \infty} g''(x_n) = g''(x_0).$$

We observe that

$$| - t^2 \cos(tx) f(t)| \leq |t^2 f(t)|$$

and we know that $t^2 f(t) \in L^1([-1, 1])$. So by the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n \to \infty} \int_{-1}^{1} -t^2 \cos(tx_n) f(t) dt = \int_{-1}^{1} \lim_{n \to \infty} -t^2 \cos(tx_n) f(t) dt$$

$$= \int_{-1}^{1} -t^2 \cos \left( t \lim_{n \to \infty} x_n \right) f(t) dt, \quad \text{since } \cos(x) \text{ is continuous}$$

$$= \int_{-1}^{1} -t^2 \cos (tx_0) f(t) dt = g''(x_0)$$

And so we have shown that $g''(x)$ both exists and continuous. Thus $g \in C^2(\mathbb{R})$. 

□
Problem 7.

a. State Fubini’s Theorem for $L^1$ functions on measure spaces. Be sure to state the hypotheses and conclusions fully and precisely.

b. For integers $m$ and $n$, define

$$f(m, n) = \begin{cases} 
1, & \text{if } m = n \\
-1, & \text{if } m = n + 1 \\
0, & \text{otherwise}
\end{cases}$$

Show that $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m, n) \neq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n)$ and explain why this inequality does not contradict Fubini’s Theorem.

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a. **Theorem (Fubini’s Theorem).** Let $X$ and $Y$ be measure spaces and let $X \times Y$ denote the maximal product measure. Suppose $f(x, y)$ is integrable on $X \times Y$. Then

(i) for almost every $x \in X$, the slice $f^x$ is integrable on $Y$, 
(ii) for almost every $y \in Y$, the slice $f^y$ is integrable on $X$, 
(iii) and

$$\int_Y \left( \int_X f(x, y) \, dx \right) \, dy = \int_X \left( \int_Y f(x, y) \, dy \right) \, dx = \int_{X \times Y} f(x, y) \, d(x \times y)$$

b. First we examine $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m, n)$. For any fixed $n$, $f(n, n) = 1$, $f(n + 1, n) = -1$ and $f(m, n) = 0$ for all other $m$. Thus $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m, n) = 0$.

Next we examine $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n)$. For every $m \neq 1$, $f(m, m) = 1$, $f(m, m - 1) = -1$ and $f(m, n) = 0$ for all other $n$. For $m = 1$, $f(m, m) = 1$ and $f(m, n) = 0$ otherwise. Thus $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) = 1$.

This does not contradict Fubini’s Theorem because $f(m, n)$ is not integrable on $\mathbb{N} \times \mathbb{N}$ in the counting measure. This is clear because

$$\sum_{m,n} |f(m, n)| \geq \sum_{m=n}^{\infty} |f(m, n)| = \sum_{1}^{\infty} 1 = \infty.$$