THE HYPERBOLIC RESTRICTED THREE-BODY PROBLEM AND
APPLICATIONS TO CELESTIAL MECHANICS

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1. Motivation

In our solar system, we have a few planets, but none as large as Jupiter. As Jupiter orbits our sun, it travels along an elliptic path in a plane at an inclination to a fixed reference plane. The magnitude of Jupiter’s size means that any effect that happens on Jupiter’s inclination or the eccentricity of its orbit will have a ripple effect on that of the other planets. It has been shown that the inclination and the eccentricity of a planet have the greatest effect on the climate of that planet. Inclination and eccentricity are two examples of what are called orbital elements, parameters which define the orbit of a planet around the sun.

Consider a star passing through our solar system “close enough” to disturb these elements of Jupiter. If a star passed within the heliosphere of our sun, and had large enough mass, what effect would it have on Jupiter? If we want to model such a situation we need to take into account three bodies: our Sun, the star passing through the system, and Jupiter. Here we have a three-body problem, a case of the famous $n$-body problem.

We want to study how Jupiter’s orbital elements change as time progresses from the past to the future. Assuming that our star doesn’t pass close enough to capture Jupiter, we hypothesize that the effect of the star could be felt while the star is nearby but disappear with the star. If these effects are large enough to change the climate on other planets, such as Earth, then we could try to see if any past climate data matches with such an event happening.

In this paper, we will first explore the Kepler problem and the restricted 3-body problem as mathematical ways to phrase the scattering problem. We will look at an example of a restricted 3-body problem to get a feel for how such a problem could be solved. We will also look at some properties of Hamiltonian systems, as we are working with one, and regularization methods as ways to deal with potential singularities. Lastly, we will utilize all this information to make some progress on our scattering problem.

2. The Kepler Problem

Before the star appears, and after the star passes through the system, we have only two bodies: Jupiter and the Sun. Before studying what happens in the system with the star, we want to look into the system of celestial bodies. In classical mechanics, the two-body problem is to determine the position and speed of two bodies interacting with each other given their masses, initial positions, and initial velocities. The gravitational two-body problem is a special case of the two-body problem in which the two bodies interact by a central force, $F$, that varies in strength as the inverse square of the distance, $r$, between them.

Let $m_1$ and $m_2$ denote the masses of the two bodies, $\mathbf{x}_1 = (x_1, y_1)$, $\mathbf{x}_2 = (x_2, y_2)$ their positions, and $r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ the distance between them. By Newton’s law of universal gravitation, the force of attraction between the particles is given by

$$F = \frac{G m_1 m_2}{r^2},$$

where $G = 6.67408 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$ is the gravitation constant. The differential equations for the gravitational two-body problem are given by

$$\begin{cases} 
    m_1 \ddot{x}_1 = F \cdot \frac{\mathbf{x}_2 - \mathbf{x}_1}{r} \\
    m_2 \ddot{x}_2 = F \cdot \frac{\mathbf{x}_1 - \mathbf{x}_2}{r},
\end{cases}
$$

(2.1)
where it is assumed that $x_1(t_0), x_2(t_0), \dot{x}_1(t_0), \dot{x}_2(t_0)$ are given at some initial time $t_0$.

### 2.1. One-Body Central Force Problem

It is possible to reduce the gravitational two-body problem above to a one-body central force problem, also known as the Kepler problem. Let $q = x_2 - x_1$ and $|q| = q$. Then, divide the first of Eqs. (2.1) by $m_1$ and the second by $m_2$ and subtract the first from the second. This yields the equation

$$\ddot{q} = -F \cdot \frac{q}{q^3}.$$  \hspace{1cm} (2.2)

Here we have what’s called the Kepler problem. Once we find out what $q$ is, we can solve for both $x_1$ and $x_2$. The original two body gravitational problem has been reduced to an equivalent one body problem. The parameter $q$ in the one body problem, called a relative coordinate, represents the distance between the “single body” and the central point. To solve (2.2), we let $\mu = G(m_1 + m_2)$ and take the resulting second order equation and turn it into a system of first order equations. We thus get the system

$$\begin{align*}
\dot{q} &= p \\
\dot{p} &= -\mu q^{-3} q.
\end{align*}$$  \hspace{1cm} (2.3)

Now to solve this, we need to consider both the conservation of angular momentum and conservation of energy. The techniques used to solve this problem can be found in many textbooks, including [12].

#### 2.1.1. Conserved Quantities

Angular momentum is the rotational analog of linear momentum. It is defined for a point particle to be the vector $q \times p$. So, we observe

$$q \times \dot{p} = -\mu q^{-3} (q \times q) = 0$$

$$\implies \frac{d}{dt} (q \times p) = q \times \dot{p} + p \times p = 0.$$  

Thus we have that $q \times p = c$, a constant vector, and we observe that angular momentum is conserved. We will refer to $c$ as the angular momentum.

In our original gravitational two-body problem (2.1), the gravitational interaction was an internal conservative force and thus we can conclude that energy in (2.3) is constant as well. We can see this mathematically by first taking the second of equations (2.3) and taking the inner product on both sides with $p$. First, note that since $q^2 = q \cdot q$, we can see by taking the derivative that $q \dot{q} = q \cdot \dot{q}$. Thus,

$$\dot{p} \cdot p = -\mu q^{-3} (q \cdot p) = -\mu q^{-3} (q \cdot \dot{q}) = -\mu q^{-3} q \dot{q} = -\mu q^{-2} dq \quad (2.4)$$

Now, if we integrate both sides of (2.4) and we find that

$$\frac{1}{2} p^2 = \mu q^{-1} + h,$$

where $h$ is a constant.

Along with angular velocity, there is one other vector which stays constant – the vector representing the eccentric axis, denoted by $e$. We will derive it below. Recall that if $r \neq 0$, then
using \( q^2 = q \cdot q \), \( q \dot{q} = q \cdot \dot{q} \), and the vector identity \((a \times b) \times c = (a \cdot c)b - (b \cdot c)a\), we get

\[
\frac{d}{dt} q = \frac{q \dot{q} - q \dot{q}}{q^2} = \frac{(q \cdot q) \dot{q} - (q \cdot q) \dot{q}}{q^2} = \frac{(q \cdot \dot{q}) \dot{q} - (q \cdot \dot{q}) q}{q^2} = \frac{(q \times \dot{q}) \times q}{q^3} = \frac{c \times q}{q^3}.
\]

Take both sides and multiply by \(-\mu\),

\[
-\mu \frac{d}{dt} q = c \times (-\mu q^{-3} q)
\]

\[
\mu \frac{d}{dt} q = \dot{p} \times c.
\]

If we integrate both sides with respect to \( t \), we get that

\[
\mu \left( e + \frac{q}{q} \right) = p \times c,
\]

where \( e \) is the constant of integration.

2.1.2. Solution of the Kepler Problem. To solve the Kepler problem, we must consider two cases from above: when \( c = 0 \) and when \( c \neq 0 \).

When \( c = 0 \), from (2.5) we get that \( \frac{q}{q} = -e \). Thus, \( e \) lies on the line of motion and has length 1.

Now let us consider what happens when \( c \neq 0 \). We know that \( q \cdot c = 0 \) and so this implies that \( e \cdot c = 0 \), that is, \( e \) and \( c \) are perpendicular and so \( e \) lies in the plane of motion. Taking the inner product of both sides of (2.5) with \( q \) gets us

\[
\mu (e \cdot q + q) = q \cdot p \times c = q \times p \cdot c = c \cdot c
\]

(2.6)

\[
\Rightarrow e \cdot q + q = \frac{c^2}{\mu}.
\]

(2.7)

Suppose \( e = 0 \), then we have that \( q = \frac{c^2}{\mu} \) and the motion is circular with uniform angular velocity. Now suppose \( e \neq 0 \). Let the fixed angle from the \( x \)-axis to \( e \) be denoted by \( \omega \). Then, let \((r, \theta)\) represent the polar coordinates of the particle from the positive \( x \)-axis. Now define \( f := \theta - \omega \), which we will call the true anomaly. If we use \( e \) as the axis of coordinates, we can represent the particle’s location with the coordinates \((r, f)\) instead. We know that \( e \cdot r = er \cos(f) \), and so applying this to the equation above, we get

\[
r = \frac{c^2/\mu}{1 + e \cos(f)}.
\]

(2.8)

Equation (2.8) is the solution to the Kepler problem. Kepler’s first law tells us that the particle moves on a conic section of eccentricity \( e \) and thus we know that (2.8) must represent a conic section. We recall that \( 0 < e < 1 \) represents an ellipse, \( e = 1 \) represents a parabola, \( e > 1 \) represents a hyperbola, and \( e = 0 \) represents a circle.
3. The Restricted 3-Body Problem

As the star gets closer to Jupiter and the Sun, we introduce a third body into the system. Now we want to work with the three-body problem. The three-body problem is the problem of determining the motions of the three bodies given an initial set of data that specifies the position, mass, and velocity of the bodies at some fixed time. If we say that the position of the masses \( m_i \) is given by \( x_i \), then we get the following three coupled second-order differential equations:

\[
\ddot{x}_1 = -Gm_2 \frac{x_1 - x_2}{|x_1 - x_2|^3} - Gm_3 \frac{x_1 - x_3}{|x_1 - x_3|^3} \\
\ddot{x}_2 = -Gm_1 \frac{x_2 - x_1}{|x_2 - x_1|^3} - Gm_3 \frac{x_2 - x_3}{|x_2 - x_3|^3} \\
\ddot{x}_3 = -Gm_1 \frac{x_3 - x_1}{|x_3 - x_1|^3} - Gm_2 \frac{x_3 - x_2}{|x_3 - x_2|^3},
\]

\[(3.1)\]

It has been shown that there is no general analytic solution for the three-body problem by algebraic expressions and integrals, but we can classify types of solutions that exist. In this paper, we are concerned with solutions near infinity. What kinds of solutions can we see when the third body moves further and further away from the primaries? These solutions can be classified by the behavior of the distance between this third body and the center of mass of the primaries. If this rate is large enough, the distance will approach infinity while the velocity approaches a limit. If this limit is zero, we call the corresponding solution parabolic. If this limit is positive, we call such a solution hyperbolic. If this rate is not large enough to overcome the attraction of the third body to the primaries, the distance will start to decrease at some point and the third mass approaches the primaries once more. Such a solution is called elliptic. An escape orbit is a solution where the distance between the primaries and the third body is bounded in backward time but goes to infinity in forward time while a capture orbit is the exact opposite - the distance between the third body and the primaries is bounded in forward time and infinite in backward time. If the distance is infinite in both forward and backward time, we have a scattering solution.

In our problem, as described in the beginning, we have that the distance of the star to the primaries, Jupiter and the Sun, starts out infinite and it comes closer, lastly going off to infinity again. Our solution is a little different from a scattering solution, seeing as one of the bodies with mass is the one that comes closer and then goes off to infinity.

The restricted three-body problem is a special case of the three-body problem where one of the bodies is said to be so much smaller than than the other two that we say it has negligible mass. So, we study the motion of this small particle moving under the influence of the gravitational attraction of the other two bodies. Thus, the equations become

\[
\begin{align*}
\ddot{x}_1 &= -Gm_2 \frac{x_1 - x_2}{|x_1 - x_2|^3} \\
\ddot{x}_2 &= -Gm_1 \frac{x_2 - x_1}{|x_2 - x_1|^3} \\
\ddot{x}_3 &= -Gm_1 \frac{x_3 - x_1}{|x_3 - x_1|^3} - Gm_2 \frac{x_3 - x_2}{|x_3 - x_2|^3},
\end{align*}
\]

where \( x_3 \) is the particle with mass 0. We quickly notice that the first two equations are just the two-body problem from before, for which we know the solution. The two primaries, the bodies with mass \( m_1 \) and \( m_2 \), have a solution that is a conic section, as we learned from the previous section. The third body, with mass \( m_3 \), simply moves in the gravitational field of the other two bodies. It does not exert influence on the paths of the primaries.
3.1. The Circular Restricted 3-Body Problem. We could solve the Kepler problem using classical mechanics but we need to be more clever with the restricted three body problem. While it has been shown that there are periodic solutions and solutions in special cases, a general solution does not exist. While the following section of this paper is not immediately relevant to the scattering problem, it provides insightful background into understanding and working with these systems.

The restricted three-body problem is often analyzed in a rotating coordinate system. In a coordinate system rotating with angular velocity \( \omega \), the two primaries are kept stationary and the third body moves about the center of mass. We can plug this solution into the third equation of the gravitational field created by \( m_1 \) and \( m_2 \) - generally an analytic solution is not possible.

3.1.1. Rotating Coordinate System. We will now move into the rotating coordinate system where \( m_1 \) and \( m_2 \) are stationary. To find our new equations, we will first consider \( x_1 \) \( \in \mathbb{C} \). We will also take the center of mass of the bodies \( m_1 \) and \( m_2 \) to be the origin, that is, let \( m_1 x_1 + m_2 x_2 = 0 \). The solution to the first two equations of system (3.1) are just circular orbits, and so they are given by

\[
x_1 = \alpha e^{i\omega t} \quad \text{and} \quad x_2 = \beta e^{i\omega t},
\]

where \( \alpha, \beta \in \mathbb{R} \). We want to rewrite the motion of the third body, described by the third equation of system (3.1) in terms of our rotating coordinates. For this, we need to see how the velocity and acceleration have changed in our new coordinate system.

### Coriolis Effect

Deflection of an object due to the Coriolis force is called the Coriolis effect.

### Sitnikov Problem

A special case of the restricted 3-body problem is the Sitnikov problem which attempts to describe the motion of three celestial bodies configured as follows:

1. The two primaries are of equal mass \( m_1 = m_2 = \frac{M}{2} > 0 \) and are moving under Newton’s Law of attraction in elliptic orbits, where the center of mass is at rest.
2. The massless body, \( m_3 = 0 \), is moving on a line \( L \) that is perpendicular to the plane of motion of the two primaries and goes through the center of mass.

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1. The Coriolis effect is an inertial force that acts on objects that are in motion relative to a rotating reference frame. Deflection of an object due to the Coriolis force is called the Coriolis effect.

2. The centripetal acceleration is the rate of change of the tangential velocity.
As we know, the massless body, or the third body, will not affect the motion of the primaries but we also notice that the symmetry of the problem makes it so that the massless body stays on $L$. To make our calculations easier, we normalize the problem so that $m_1 + m_2 = M = 1$, set the periodic of the elliptic orbits of the primaries to $2\pi$, and let the gravitational constant be 1.

We will let $z$ denote the coordinates of the massless body on the line $L$ so that $z = 0$ corresponds to the center of mass and $t$ will be our time. The ODE describing this problem is

$$\ddot{z} = -\frac{z}{(z^2 + r(t)^2)^{3/2}}$$

where $r(t) = r(t + 2\pi) > 0$, which is the distance of one of the primaries from the center of mass, an even function of $t$. Let $e$ represent the eccentricity, then we know that for small $e > 0$,

$$r(t) = \frac{1}{2} (1 - e \cos(t)) + O(e^2).$$

We want to study the path of our third body and here we specifically look at solutions near the critical escape velocity as this is where the distance between the third body and the primaries goes to infinity.

The solution orbits of this problem were studied by Sitnikov and Alexeev. Sitnikov proved the existence of oscillation, capture, and escape orbits.

3.2.1. Solution of the Sitnikov problem. In his 1973 monograph, Moser gave a detailed, geometric argument to show the existence of these orbits.

To solve this problem, we will be associating solutions to sequences of integers. So consider the solution $z(t)$ that has infinitely many zeroes at $t_k$, $k \in \mathbb{Z}$ which are ordered such that $t_k < t_{k+1}$, so $z(t_k) = 0$ for all $k \in \mathbb{Z}$. Then, consider the integers

$$s_k = \left\lfloor \frac{t_{k+1} - t_k}{2\pi} \right\rfloor,$$

where $s_k$ measures the number of completed revolutions of the primary bodies between any two zeros of $z(t)$, that is, between any two times the third body passes through the center of mass. So, any solution can be associated to a doubly infinite sequence of integers in this way. This gives rise to a theorem:

**Theorem 1.** Given a sufficiently small eccentricity, $e > 0$, there exists an integer $m = m(e)$ such that any sequence $s = \ldots s_{-1} s_0 s_1 s_2 \ldots s_k \ldots$, $k \in \mathbb{Z}$, with $s_k \geq m$, corresponds to a solution of the differential equation (3.2). [10]
These \( s_k \) can be chosen independently. That is, for any given sequence \( s_k \), there exists a corresponding solution. If one chooses an unbounded sequence then the solution will be unbounded but still have infinitely many zeros. Choosing a periodic sequence allows us to find infinitely many periodic orbits as well. We can also take the smallness assumption on \( e \) away, as the statement holds true for \( 0 < e < 1 \) except on a discrete set of values.

To prove this theorem, Moser utilized symbolic dynamics. To do this, he constructed a collection of “boxes” and flow-defined Poincaré maps. These maps stretch the boxes on top of one another in a certain way. The boxes are connected by the Poincaré map and any sequence, \( s_k \), will correspond to certain boxes. There exists a solution of this problem that realizes the sequence in the sense that the solution passes through the boxes in the given order. The use of such symbolic dynamics is only possible if the boxes, which are manifolds, intersect transversally. Moser takes care to show this in \([10]\).

A part of Moser’s work incorporated a coordinate system introduced by McGehee \([7]\) which pastes a two dimensional boundary manifold at infinity to the three-dimensional constant energy manifold. At infinity, there is a single periodic orbit to which all parabolic orbits converge and McGehee showed in particular that the parabolic solutions form the analytic stable and unstable manifolds of this periodic orbit \([7]\).

The out of plane approach of the massless body is of particular interest to us in the Sitnikov problem. We will later explore what happens when our star in the scattering problem approaches out of the plane of the two other bodies, the Sun and Jupiter. The solution method utilizing manifolds at infinity could also be useful to us as we attempt to work on the scattering problem.

### 4. Hamiltonian Systems

A Hamiltonian system is a dynamical system that is completely described by a scalar function \( H(q, p, t) \), known as the Hamiltonian, where \( q, p \in \mathbb{R}^n \). The differential equations for the system are given by the Hamiltonian as follows:

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.
\]

(4.1)

Here, \((q(t), p(t))\) is the solution of the IVP defined by the above equations, known as Hamilton’s equations, and an initial condition \((q(t_0), p(t_0)) = (q_0, p_0)\). If the Hamiltonian does not explicitly depend on time, then the Hamiltonian is a constant of motion equaling the total energy of the system.

The problems we have described thus far can be written as Hamiltonian systems. Consider the Kepler problem once more. This is a Hamiltonian system, with

\[
H(q, p) = \frac{p^2}{2} - \frac{\mu}{q}.
\]

(4.2)

This equation matches the equation of the energy derived earlier for the Kepler problem.

#### 4.1. Symplectic Transformations

A differentiable map \( g : U \to \mathbb{R}^{2n} \), where \( U \subset \mathbb{R}^{2n} \) is an open set, is called symplectic if the Jacobian matrix \( Dg(p, q) \) is everywhere symplectic, that is, if

\[
Dg(p, q)^{T} J Dg(p, q) = J \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]
where $I$ is the identity matrix of dimension $n$. We can write a Hamiltonian system in the form

$$\dot{y} = J^{-1} \nabla H(y),$$

where $y = (p, q)$ and $\nabla H(y) = DH(y)^T$ [8]. We recall now that the flow $\varphi_t : U \to \mathbb{R}^{2n}$ of a Hamiltonian system is a map which moves the solution forward by time $t$ and depends on $t, p_0$, and $q_0$. In 1892, Poincaré proved the following theorem which tells us that Hamiltonian systems are symplectic [11].

**Theorem 2.** If $H(p, q)$ is a twice continuously differentiable function on $U \subset \mathbb{R}^{2n}$, then for each fixed $t$, the flow $\varphi_t$ is a symplectic transformation wherever it is defined.

One important result of the fact that Hamiltonian systems have a symplectic structure is that an infinitesimal phase-space volume is preserved. Consider the change of variables $y \to \xi = \Xi(y, t)$

Then we say that the transformation is symplectic if and only if

$$\frac{\partial \Xi}{\partial y} J \frac{\partial \Xi^T}{\partial y} = J.$$

The determinant of a symplectic matrix is $+1$ and therefore the transformation is orientation and volume-preserving. Thus, when using transformations on Hamiltonian systems, it is important to use symplectic transformations so that the symplectic structure is maintained in our new system.

### 5. Regularizations

One thing we want to keep in mind as we’re working with the Kepler problem or the restricted 3-body problem is collisions. A collision set,

$$\Delta = \{ q \mid q_i = q_j \text{ for some } i \neq j \},$$

is the set of points where more than 1 body occupies the same position. A singularity $t_0$ of the 3-body problem is a collision singularity when $q(t)$ approaches a specific point of $\Delta$ as $t \to t_0$. When collisions occur, Newton’s Law of Gravity cease to make sense. A regularization is a transformation $(q, p) \mapsto (u, w)$ where $q = 0$ corresponds to some $u = u_0$ and $|w(s)| \to w_0$ as $s \to s_0$. We can extend collision solutions smoothly as a function of our new time variables through $s = s_0$. Historically, regularization was developed to deal with these singularities in Kepler motion and for describing the collisions of two point masses.

In our problem, we are concerned with the effects that scattering might have on Jupiter. Jupiter’s orbit could get exceedingly eccentric and have near collisions with the sun. We want to make sure we avoid such cases and can use regularization to make that happen. Below we describe a few established regularization methods.

#### 5.1. Levi-Civita Regularization.

The first regularization method we’ll discuss is the Levi-Civita regularization. To apply this transformation, we will use complex notation throughout, i.e. instead of the vector $q = (q_1, q_2) \in \mathbb{R}^2$ we use the corresponding complex coordinate $q = q_1 + iq_2 \in \mathbb{C}$. The Levi-Civita transformation is [5]

$$q = 2z^2,$$

$$p = \frac{w}{z}.$$

where $q, p, z, w \in \mathbb{C}$. Now, the Hamiltonian for the Kepler problem is given by [4][2]. Changing the Hamiltonian, we get that

$$H \left( 2z^2, \frac{w}{z} \right) = \frac{1}{2} \left| \frac{w}{z} \right| - \frac{\mu}{2|z|^2} = \frac{1}{2} \left| \frac{w}{z} \right| (|w|^2 = \mu).$$
Consider the energy surface where \( H(q,p) = -c^2 \), where \( c > 0 \). This corresponds to solutions being ellipses. Then,

\[
K(z, w) = 2|z|^2 \left( \left( 2z^2, \frac{w}{2} \right) + c^2 \right) = |w|^2 - \mu + 2|z|^2 c^2 = 0
\]

is the Hamiltonian for the system

\[
\begin{align*}
z' &= Kw = w \\
w' &= -Kz = -2c^2 z,
\end{align*}
\]

where we have rescaled the time variable so that \( z' = \frac{dz}{d\tau} \) and \( dt = 2|z|^2 d\tau \).

This transformation removes the singular point at the origin. The singularities of the transformed equations now lies at infinity and at the images of the body \( m_1 \) in the \( q, p \)-plane.

### 5.2. Moser’s Regularization

In 1970, Moser showed that the solutions to the Kepler problem with negative energy are equivalent to the geodesic flow on the sphere \([9]\). Moser showed this by beginning with equations whose solutions are the geodesic flow and transforming this problem into the Kepler problem. Moser proved this for the \( n \)-dimensional case, but here, to simplify, we will show it for \( n = 2 \). The following proof is taken from \([9]\).

First, we take \( \xi = (\xi_1, \xi_2, \xi_3) \) be a real vector so that \( |\xi| = 1 \) represents \( S^2 \). The geodesic flow is the solution of the variational problem

\[
\delta \int |\xi'(s)|^2 ds = 0,
\]

where \( \xi(s) \) satisfies the property \( |\xi(s)| = 1 \). The Euler equations to this problem are

\[
\xi'' + \lambda^2 \xi = 0,
\]

where \( \lambda = |\xi'(s)| \) is a real constant. Now, we will write this second order problem as a system of first order problems by letting \( \eta = \xi' \). Then,

\[
\begin{align*}
\xi' &= \eta \\
\eta' &= -|\eta|^2 \xi,
\end{align*}
\]

where we still have \( |\xi| = 1 \) and we also have \( \langle \xi, \eta \rangle = \sum_{\nu=1}^{3} \xi_{\nu} \eta_{\nu} = 0 \). This system can be written as a Hamiltonian system, with Hamiltonian

\[
\Phi(\xi, \eta) = \frac{1}{2} |\xi|^2 |\eta|^2
\]

so

\[
\eta' = \Phi_\eta(\xi, \eta) \quad \text{and} \quad \xi' = -\Phi_\xi(\xi, \eta)
\]
on \( T(S^2) \), the tangent bundle of \( S^2 \).

Now, to describe the flow in Euclidean space, we will use the stereographic projection -

\[
x_k = \frac{\xi_k}{1 - \xi_0}, \quad \text{where} \ k = 1, 2, 3.
\]

This maps the punctured sphere, which we will denote as \( \hat{S}^2 \), onto 3-dimensional Euclidean space. We wish to extend this map with

\[
y_k = g_k(\xi, \eta), \quad \text{where} \ k = 1, 2, 3
\]
to a mapping of the tangent bundle of \( \hat{S}^2 \), \( T(\hat{S}^2) \), into the tangent bundle of \( \mathbb{R}^3 \), \( T(\mathbb{R}^3) = \mathbb{R}^{2,3} = \mathbb{R}^6 \). To ensure that the symplectic structure holds, we demand that

\[
\sum_{\nu=0}^{n} \eta_{\nu} \, d\xi_{\nu} = \sum_{k=0}^{n} y_{k} \, dx_{k}.
\]

Such an extension exists and is given by

\[
y_{k} = \eta_{k}(1 - \xi_{0}) + \xi_{k} \eta_{0}, \quad \text{where } k = 1, 2, 3.
\]

This extension is also unique. So, this map given by (5.2) and (5.4),

\[
(\xi, \eta) \mapsto (x, y), \quad \text{maps } T(\hat{S}^2) \to \mathbb{R}^6.
\]

We will discuss how we came upon (5.5) shortly, but for now, we attempt to construct the inverse map. We recall that \( |\xi| = 1 \). Then,

\[
|\xi|^{2} = \sum_{k=1}^{2} x_{k}^{2} = \sum_{k=1}^{2} \frac{\xi_{k}^{2}}{(1 - \xi_{0})^{2}} = \frac{1}{(1 - \xi_{0})^{2}} \cdot (1 - \xi_{0}^{2}) = \frac{1 + \xi_{0}}{1 - \xi_{0}},
\]

which implies that

\[
\xi_{0} = \frac{|\xi|^{2} - 1}{|\xi|^{2} + 1} \quad \text{and} \quad \xi_{k} = x_{k}(1 - \xi_{0}), \quad k = 1, 2, \ldots, n
\]

\[
= x_{k} \left( 1 - \frac{|\xi|^{2} - 1}{|\xi|^{2} + 1} \right)
\]

\[
= x_{k} \left( \frac{|\xi|^{2} + 1 - |\xi|^{2} + 1}{|\xi|^{2} + 1} \right)
\]

\[
= \frac{2x_{k}}{|\xi|^{2} + 1}.
\]

Therefore we have that the inverse is given by

\[
(5.6) \quad \xi_{0} = \frac{|\xi|^{2} - 1}{|\xi|^{2} + 1} \quad \text{and} \quad \xi_{k} = \frac{2x_{k}}{|\xi|^{2} + 1}.
\]

Using (5.4) and the fact that \( (\xi, \eta) = 0 \), we find

\[
(x, y) = \sum_{k=1}^{2} x_{k} y_{k} = \sum_{k=1}^{2} \frac{\xi_{k}}{1 - \xi_{0}} \left( \eta_{k}(1 - \xi_{0}) + \xi_{k} \eta_{0} \right)
\]

\[
= \sum_{k=1}^{2} \left( \xi_{k} \eta_{k} + \frac{\xi_{0}^{2} \eta_{k}}{1 - \xi_{0}} \right)
\]

\[
= -\xi_{0} \eta_{0} + \frac{\eta_{0}}{1 - \xi_{0}} (1 - \xi_{0}^{2})
\]

\[
= -\xi_{0} \eta_{0} + (1 + \xi_{0}) \eta_{0}
\]

\[
= \eta_{0}.
\]

We wanted to find this identity so we can find the inverse of \( y_{k} \). Now, we will solve for \( \eta_{k} \) from (5.5):

\[
y_{k} = \eta_{k}(1 - \xi_{0}) + \xi_{k} \eta_{0} = \eta_{k} \left( 1 - \frac{|\xi|^{2} - 1}{|\xi|^{2} + 1} \right) + \frac{2x_{k}}{|\xi|^{2} + 1}(x, y) = \frac{2}{|\xi|^{2} + 1} (\eta_{k} + x_{k}(x, y)).
\]

Using this, we find that

\[
(5.7) \quad \eta_{0} = (x, y) \quad \text{and} \quad \eta_{k} = \frac{|x|^{2} + 1}{2} y_{k} - (x, y) x_{k}.
\]
Before moving forward, we quickly check that for any \((x, y)\) that the properties \(|\xi| = 1\) and \((\xi, \eta) = 0\) hold for (5.5) and (5.6):

\[
|\xi|^2 = \xi_0^2 + \sum_{\nu=1}^n \xi_{\nu}^2 = \left(\frac{|x|^2 - 1}{|x|^2 + 1}\right)^2 + \sum_{\nu=1}^n \left(\frac{2x_\nu}{|x|^2 + 1}\right)^2 = \left(\frac{|x|^2 - 1}{|x|^2 + 1}\right)^2 + \frac{4}{(|x|^2+1)^2}|x|^2 = \frac{|x|^4 - 2|x|^2 + 1 + 4|x|^2}{(|x|^2+1)^2} = 1
\]

\[
(\xi, \eta) = \xi_0 \eta_0 + \sum_{\nu=1}^n \xi_\nu \eta_\nu = \frac{|x|^2 - 1}{|x|^2 + 1}(x, y) + \sum_{\nu=1}^n \frac{2x_\nu}{|x|^2 + 1} \left(\frac{|x|^2 + 1}{2} y_\nu - (x, y)x_\nu\right) = \frac{|x|^2 - 1}{|x|^2 + 1}(x, y) + \sum_{\nu=1}^n x_\nu y_\nu - \frac{2}{|x|^2 + 1} \sum_{\nu=1}^n x_\nu^2 = \frac{|x|^2 (x, y) - (x, y) + |x|^2 (x, y) - 2|x|^2 (x, y)}{|x|^2 + 1} = 0.
\]

Thus, our transformation successfully maps \(\mathbb{R}^6 \to T(S^2)\). One more important equality we want to establish is \(|\eta|\) in terms of \(x\) and \(y\) as this will be useful when transforming the Hamiltonian

\[
|\eta|^2 = \eta_0^2 + \sum_{\nu=1}^2 \eta_\nu^2 = (x, y)^2 + \sum_{\nu=1}^2 \left(\frac{|x|^2 + 1}{2} y_\nu - (x, y)x_\nu\right)^2 = (x, y)^2 + \sum_{\nu=1}^2 \left(\frac{|x|^2 + 1}{2} y_\nu^2 - \frac{|x|^2 + 1}{2} (x, y)x_\nu y_\nu + (x, y)^2 x_\nu^2\right) = (x, y)^2 + \left(\frac{|x|^2 + 1}{2}\right)^2 |y|^2 - (|x|^2 + 1)(x, y)^2 + (x, y)^2 |x|^2,
\]

which tells us that

\[
(5.8) \quad |\eta| = \left(\frac{|x|^2 + 1}{2}\right) |y|.
\]

Now we quickly want to work out how (5.5) was found. We use the conformal nature of the stereographic projection and get

\[
|d\xi|^2 = \frac{4}{(|x|^2+1)^2}|dx|^2.
\]

Let \(\xi = \psi(x)\) and the Jacobian \(n \times (n + 1)\) matrix will be given by \(\psi_x(x)\). Then, using (5.6) we find that

\[
\psi^T_x(x) \psi_x(x) = \frac{4}{|x|^2 + 1} I_n,
\]

where $I_n$ represents the $n \times n$ identity matrix. Using the relation
\[(\eta, d\xi) = (\eta, \psi_x dx) = (\psi^T_x \eta, dx),\]
we see that it would be effective to let $\eta = \gamma \psi_x y$, where $\gamma$ is an undetermined scalar factor. So now, we want to figure out what this $\gamma$ should be. If we plug this $\eta$ into (5.4), we find that it is satisfied when
\[
\gamma = \left(\frac{|x|^2 + 1}{4}\right).
\]
It is good to check here that this value of $\gamma$ still keeps $\eta$ and $\xi$ perpendicular. Thus, we get
\[
\eta = \left(\frac{|x|^2 + 1}{4}\right) \psi_x y.
\]
This defines (5.7) which we can use to get (5.5) as well.

Since we constructed our transformation to preserve the Hamiltonian structure, we can find our new differential equations easily by transforming the Hamiltonian -
\[
F(x, y) = \Phi(\xi, \eta) = \frac{1}{2} |\xi|^2 |\eta|^2 = \left(\frac{|x|^2 + 1}{8}\right) |y|^2.
\]
Thus, our new system is given by
\[
(5.9) \quad x' = F_y, \quad y' = -F_x.
\]
We will now consider the case of geodesics with unit velocity, $|\eta| = 1$, which corresponds to $\Phi(\xi, \eta) = \frac{1}{2}$. This means we are looking at solutions with $F = \frac{1}{2}$. We replace $F$ by a new function $G = u(F)$ where $u'(\frac{1}{2}) = 1$ because on $F = \frac{1}{2}$ the gradients of $F$ and $u(F)$ are the same. So, we let
\[
G = u(F) = \sqrt{2F} - 1 = \frac{|x|^2 + 1}{2} |y| - 1.
\]
Now on $F = \frac{1}{2}$, (5.8) becomes $x' = G_y, \ y' = -G_x$ on $G = 0$. Now, we change the independent variable from $s$ to $t$ where
\[
t = \int |y| \, ds,
\]
which results in the new system
\[
x = |y|^{-1} G_y, \quad y = -|y|^{-1} G_x
\]
on $G = 0$. Now, we use Poincaré’s trick to find a new system with a Hamiltonian $H$ that satisfies
\[
H_y = |y|^{-1} G_y \quad \text{and} \quad H_x = |y|^{-1} G_x.
\]
Changing the variables in our Hamiltonian gets us that
\[
H = |y|^{-1} G - \frac{1}{2} = |y|^{-1} \left(\sqrt{2F} - 1\right) - \frac{1}{2} = \frac{1}{2} |x|^2 - \frac{1}{|y|}.
\]
Let $p = -x$ and $q = y$ and we get the Kepler problem on the energy surface $-\frac{1}{2}$:
\[
H = \frac{1}{2} |p|^2 - \frac{1}{|q|} = -\frac{1}{2}.
\]
Lastly, we now include the excluded north pole, compactifying the energy surface. This is equivalent to $T(S^2)$ and geodesics through the north pole become collision orbits.
5.3. Belbruno’s Regularization. Utilizing Moser’s methods, in 1975 Belbruno showed a very similar result but using the geodesic flow on a hyperboloid [2]. In terms of Kepler motion, Belbruno looked at the case where the Hamiltonian $H > 0$ while Moser considered the case where $H < 0$. As the method of proof is very similar, we will leave it out. The proof can be found in [2].

Let $\mathcal{H}_\pm$ represent the sheets of the hyperboloid given by $-\xi_0^2 + \xi_1^2 + \xi_2^2 = -1$, with $+$ representing the upper sheet and $-$ representing the lower sheet. Before we work with this hyperboloid, we want to remove the points $(\pm 1, 0, 0)$, similarly to how we removed the north pole in the previous section. These points represent collision orbits. The Kepler flow is regularized by restoring $(\pm 1, 0, 0)$ at the end.

Once more, we take $\xi = (\xi_0, \xi_1, \xi_2)$ to be a real vector and this time we let $\langle \xi, \eta \rangle = -\xi_0 \eta_0 + \xi_1 \eta_1 + \xi_2 \eta_2$. Now we let $\mathcal{H}_\pm$ be the hyperboloid as described above embedded in a Lorentz space $\mathbb{L}^3$, a vector space with the inner product given above. Belbruno uses a variational problem once more to define the geodesic flow, then shows that the Hamiltonian of this system is the same as the Hamiltonian of the Kepler problem using some coordinate transformations.

6. The Scattering Problem

Now that we have put together a set of tools, let us look at the problem at hand again. To make our problem into a restricted three-body problem, we need to let one of the bodies have mass 0. Thus, we’ll let our sun and the passing star have masses $m_1$ and $m_2$ respectively, letting Jupiter be our massless object orbiting our sun with mass $m_3 = 0$. To visualize the problem in 2D, we’ll fix our sun at the origin, with Jupiter orbiting around it on an ellipse. The star will pass by on a hyperbolic orbit with the closest approach at time $t = 0$ on the positive $x$-axis. Recall that this sort of solution is called a scattering solution. So the goal is to understand the scattering map taking the initial state of the planet Jupiter at the time $t = -\infty$ to the final state at the time $t = \infty$. The idea is that the planet will behave approximately as the solution to the Kepler problem before and after the star passes.

6.1. Computing Orbital Elements. As mentioned, our goal is to study the orbital elements of the third body as the star passes through the system. A way to compute the orbital elements is to use the Laplace vector, also called the eccentricity vector. The Laplace vector is a vector used to describe the shape and orientation of the orbit of one astronomical body around another. In the Kepler problem, recall from Section 2.1.1 that the angular momentum, $\mathbf{e} = \mathbf{q} \times \mathbf{p}$ is conserved. Consider the planar problem, where $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$, then we use the planar cross product and assume the convention that $\mathbf{e} = q_1 p_2 - q_2 p_1$. The angular momentum is a constant of motion, a function that is constant along an orbit. Let the eccentricity vector have coordinate $\mathbf{e} = (\alpha, \beta)$. Then, using (2.7), we see that

$$\alpha q_1 + \beta q_2 = q - c^2 \mu,$$

where $q = |\mathbf{q}|$. From this, we can see that the eccentricity vector’s coordinates are $[11]$

$$\mathbf{e} = (\alpha, \beta) = \left( \frac{q_1}{q} - \frac{c}{\mu} p_2, \frac{q_2}{q} + \frac{c}{\mu} p_1 \right).$$

The norm of the eccentricity vector, $e = |\mathbf{e}|^2 = \alpha^2 + \beta^2$, gives us the instantaneous eccentricity of the orbit given by $\mathbf{q}(t)$. We can also find the semi-major axis, $a$, using $\mathbf{e}$, as follows:

$$a = \frac{4c^2}{\mu(1 - e^2)}.$$
6.2. **Sorokovich.** In his paper from 1982, A.B. Sorokovich worked on the planar, singly averaged, hyperbolic, restricted three-body problem where the perturbing body, our body of mass \( m_2 \), is on a path with small hyperbolic eccentricity \([13]\). The general solution of the linearized system of differential equations of perturbed motion in the single averaged problem is given in the form of a series. Sorokovich studied the effects that \( m_2 \) have on the orbital elements of the perturbed body’s motion. He gave results on the eccentricity and the argument of the pericenter, which is the angle from the body’s ascending node to its periapsis.

His results suggest that the stability of the unperturbed motion holds when the orbital eccentricity of the path of the body of mass \( m_3 = 0 \) is small and it remains close to the central body. The exact and singly averaged equations were integrated numerically. Sorokovich’s work suggests that while the eccentricity of the orbit of the perturbed body changes as the perturbing body passes close by, it returns back to its original state eventually. More interestingly, we notice that the argument of the pericenter actually changes by a small amount as the star passes and does not return to its original state. The argument of the pericenter actually increases as the star passes by the orbiting planet.

6.3. **Current and Future Work.** We are interested in solutions where \( m_3 \) stays near \( m_1 \) and so it is useful to introduce a coordinate system relative to \( m_1 \). So, we’ll let

\[
Q = x_2 - x_1 \quad \text{and} \quad q = x_3 - x_1.
\]

Now, we can calculate \( \dot{Q} \) and \( \dot{q} \) by subtracting equations (3.1) as necessary and letting \( G = 1 \). We get the equations

\[
\ddot{Q} = -\frac{(m_1 + m_2)Q}{|Q|^3}, \tag{6.1}
\]

\[
\ddot{q} = -\frac{m_1 q}{|q|^3} - \frac{m_2 (q - Q)}{|q - Q|^3} - \frac{m_2 Q}{|Q|^3}, \tag{6.2}
\]

where we notice that \( r_{12} = r_{21} = |Q|, r_{13} = r_{31} = |q|, \) and \( r_{23} = r_{32} = |q - Q| \). \( Q \) is the Kepler problem with mass \( m_1 + m_2 \) and thus we can write out the solution explicitly. This describes the relative motion of the two stars, and so we want to choose a hyperbolic solution. Recall that we said the closest approach will be at \( t = 0 \) and on the positive \( x \)-axis. Such a solution can be characterized by two parameters \( a > 0 \) and \( e > 1 \). There is no simple formula for the solution \( Q(t) \), but we define both \( Q \) and \( t \) in terms of another parameter, the hyperbolic anomaly, given by \( h \). In polar coordinates, the following equations characterize the solution \([3]\):

\[
Q = R \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \tag{6.3}
\]

\[
R(h) = a\left(e\cosh(h) - 1\right), \tag{6.4}
\]

\[
\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{e + 1}{e - 1}\tanh\left(\frac{h}{2}\right)}, \tag{6.5}
\]

\[
e\sinh(h) - h = \sqrt{\frac{m_1 + m_2}{a^3}} t. \tag{6.6}
\]

Given a time \( t \), from equation \([6.6]\), we can numerically find the value of \( h \). That is, given a \( t_0 \), we solve \([6.6]\) for \( h \) and find

\[
h = \sinh^{-1}\left(\frac{1}{e}\left(h + \sqrt{\frac{m_1 + m_2}{a^3} t_0}\right)\right).
\]
This function is a contraction and therefore we can find \( h \) by iterating. Once we find \( h(t) \), we can use the fact that if we let
\[
\tau = \tan \left( \frac{\theta}{2} \right)
\]
then
\[
\cos(\theta) = \frac{1 - \tau^2}{1 + \tau^2} \quad \text{and} \quad \sin(\theta) = \frac{2\tau}{1 + \tau^2}
\]
to find \( \cos(\theta) \) and \( \sin(\theta) \). We know \( R(h) \) since we found \( h \) already, and thus we can plug this all in to (5.5) to numerically find \( Q(t) \in \mathbb{R}^2 \).

Notice that if \( t = 0 \), then \( h = 0, \theta = 0, R = a(e - 1) \) and so \( Q(0) = (a(e - 1), 0) \), which is the closest approach. This is a symmetric problem about \( t = 0 \), so we note that if we can figure out what happens from \([0, T], T > 0\), then the same thing happens in backward time on the interval \([-T, 0]\).

So, we can assume \( Q(t) \) is known and the dynamics of the small planet is given by the time-dependent differential equation (6.1). We can rewrite this second order equation as a first-order system
\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= -\frac{m_1 q}{|q|^3} - \frac{m_2 (q - Q)}{|q - Q|^3} - \frac{m_2 Q}{|Q|^3}
\end{align*}
\]
where \((q, p) \in \mathbb{R}^4\). (6.7) is a time dependent Hamiltonian system with
\[
H(q, p, t) = \frac{1}{2} |p|^2 - U(q, t),
\]
where
\[
U(q, t) = \frac{m_1}{|q|} + \frac{m_2}{|q - Q|} + \frac{m_2}{|Q|}.
\]

6.3.1. Regularization. Another possible way to deal with (6.2) would be to use a regularization. We would change the time scale and use \( h \) as our new time variable. Then we’d regularize the collision at \( q = 0 \) to avoid numerical difficulties. Let
\[
F(q, t) = \frac{m_2 (q - Q)}{|q + Q|^3} - \frac{m_2 Q}{|Q|^3}.
\]

Then, we can rewrite equations (6.7) as
\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= -\frac{m_1 q}{|q|^3} - F(q, t)
\end{align*}
\]
where \((q, p) \in \mathbb{R}^2 \) and \( F(q, t) \in \mathbb{R}^2 \) is a vector-valued perturbing function. Now, we’d like to use the Levi-Civita transformation. Instead of changing the Hamiltonian, let us change the equations, via
\[
\dot{q} = 4z \dot{z} = p = \frac{w}{z} \implies \dot{z} = \frac{w}{4|z|^2}.
\]

Now, we want to rescale time. Let \( s \) be our new time variable such that
\[
\frac{dt}{ds} = |q| = 2|z|^2.
\]

So now, we can write our ODE in terms of derivatives with respect to \( s \) instead of \( t \), where
\[
z' = \frac{dz}{ds}.
\]

\[
z' = 2|z|^2 \dot{z} = \frac{w}{2}.
\]
Now we need to find \( w' \). For this, we will use the fact that \( w = p^z \) and one can show that
\[
 w' = Kz - 2|z|^2zF(2z^2, t),
\]
where \( K \) represents the Kepler energy of \((q, p)\), that is
\[
 K = K(z, w) = \frac{|p|^2}{2} - \frac{m_1}{|q|} = \frac{|w|^2}{2|z|^2} - \frac{m_1}{2|z|^2}.
\]
We want to make sure we think of \( K \) and \( t \) in terms of \( s \) as well, so we can find differential equations for this as well. Putting all the equations together, we get a regularized system of ODEs
\[
\begin{align*}
 z' &= \frac{w}{2} \\
 w' &= Kz - 2|z|^2zF(2z^2, t) \\
 K' &= -2F(2z^2, t) \cdot zw \\
 t' &= 2|z|^2.
\end{align*}
\]
(6.10)
where we view the complex number \( zw \) as a vector in \( \mathbb{R}^2 \). Note that \( K, t \in \mathbb{R} \) so we have a system with six variables. These are all related by the definition of \( K \):
\[
|w|^2 - 2K|z|^2 = m_1.
\]
It’s important to note that if initial conditions are chosen so that the above equation holds, it should continue to hold for all time.

We can also instead use \( h = h(s) \) as our variable instead of \( t \). The function \( F \) can be thought of as \( F(q, h) \). We’ll replace the last equation for \( t' \) instead with an equation of \( h' \) and we get the system of regularized equations
\[
\begin{align*}
 z' &= \frac{w}{2} \\
 w' &= Kz - 2|z|^2zF(2z^2, h) \\
 K' &= -2F(2z^2, h) \cdot zw \\
 h' &= \frac{2|z|^2}{e \cosh(h) - 1}.
\end{align*}
\]
(6.11)
We use Mathematica to solve these equations and use the Lagrange vector (eccentricity vector) to plot a few orbital elements for various values of \( m_1 \) and \( m_2 \), making sure that \( m_1 + m_2 = 1 \). If we let \( m_1 = m_2 = 0.5 \), then below we see a parametric plot of the path of the third body in blue below with the path of the star in red. The second image below is a zoomed in version of the first image.

Figure 2. Path of Jupiter, where \( m_1 = m_2 = 0.5 \)
We use the Lagrange vector to calculate the eccentricity and the semi-major axis of the third body as time progresses. We can plot those as well using Mathematica. For the following plots, we see a zoomed in version along with the regular view.

\[ m_1 = m_2 = 0.5 \]

\[ m_1 = 0.7 \text{ and } m_2 = 0.3 \]

The above scenario has \( m_1 = m_2 \), which may not be likely. We want to increase \( m_1 \) and decrease \( m_2 \) to see how our third body is affected. Below we let \( m_1 = 0.7 \) and \( m_2 = 0.3 \) to see how this changes the situation.

\[ m_1 = 0.9 \text{ and } m_2 = 0.1 \]

We see that the path of our third body stays closer to the original path when we decrease the mass of the passing star, and luckily, this matches up with our intuition. We see similar results in the eccentricity and the semi-major axis where the eccentricity and semi-major axis have fewer changes. Our intuition is confirmed when we study the images created when \( m_1 = 0.9 \) and \( m_2 = 0.1 \) below.
6.3.2. Ideas Going Forward. While originally we were worried about having to use a symplectic integrator due to the time-dependent Hamiltonian, we have found that our regularizations and view of the problem have resulted in meaningful results from Mathematica without such an integrator. We would like to move on to working in 3-dimensions by first moving the star out of the plane to make it look similar to the Sitnikov problem. However, in the Sitnikov problem it is the massless object that is out of the plane, not the object with mass so the solution methods may not translate. We also cannot use the Levi-Civita regularization in 3 dimensions so we may have to explore using the Moser regularization or perhaps another change of coordinates we will have to come up with ourselves.
References