1. **Flows and Vector Fields**

**Definition 1.1.** A flow $\varphi_t : \mathbb{R} \times X \to X$ is a diffeomorphism with the properties

(i) $\varphi_t \circ \varphi_s = \varphi_{t+s}$

(ii) $\varphi_0 = id$

**Definition 1.2.** A local flow is a flow on an open neighborhood $U = \varphi (\{0\} \times X) \subset \mathbb{R} \times X$ and has property (i) when both sides are defined (i.e. $\varphi_t(x), \varphi_s(x), \varphi_{t+s}(x) \in U$).

**Exercise 1.3.** How do you go between a flow and a vector field? 

*Answer:* Given a flow, $\varphi_t$, the vector field associated with the flow is 

$$f(x) = \left. \frac{d}{dt} (\varphi_t(x)) \right|_{t=0}.$$ 

**Remark.** To go from a vector field to a flow requires more work!

**Proposition 1.4.** $x(t) = \varphi_t(x_0)$ is a solution to 

$$\begin{cases} 
\frac{d}{dt} x(t) = f(x(t)) \\
x(0) = x_0 
\end{cases}$$

for all $t$, where $f$ is the vector field associated with $\varphi_t(x)$.

*Proof.* We want to show that 

$$\frac{d}{dt} (\varphi_t(x_0)) \bigg|_{t=t_0} = f(\varphi_{t_0}(x_0)).$$

So, we first see that 

$$\frac{d}{dt} (\varphi_t(x_0)) \bigg|_{t=t_0} = \frac{d}{dt} (\varphi_{t-t_0} \circ \varphi_{t_0}(x_0)) \bigg|_{t=t_0}.$$ 

Now, we change variables. Let $\tau = t - t_0$. Then, when $t = t_0$, $\tau = 0$ and for any function $u$, 

$$\frac{du}{dt} = \frac{du}{d\tau} \cdot \frac{d\tau}{dt} = \frac{du}{d\tau}.$$ 

So, then we see that 

$$\frac{d}{dt} (\varphi_t(x_0)) \bigg|_{t=t_0} = \frac{d}{d\tau} (\varphi_{t-t_0} \circ \varphi_{t_0}(x_0)) \bigg|_{\tau=0}.$$ 

where the last equal sign follows by definition. \qed
Exercise 1.5. Give an example of a global flow.

Answer:
\[ \varphi_t(x) = e^{At}x \implies \frac{d}{dt} \bigg|_{t=0} e^{At}x = Ae^{At}x \bigg|_{t=0} = Ax. \]

Associated Vector Field: \( f(x) = Ax \)

ODE: \( \dot{x}(t) = Ax \)

Exercise 1.6. Given an example of a local flow.

Answer:
\[ \varphi_t(x) = \begin{cases} 
\frac{1}{x} - t & \text{where } x \neq 0 \\
\frac{1}{x} & \text{where } x = 0 
\end{cases} \]

Associated Vector Field: \( f(x) = x^2 \)

ODE: \( \dot{x}(t) = x^2(t) \)

Note that this is is a local flow because it is not defined at \( t = \frac{1}{x} \).
2. Existence and Uniqueness of ODEs

**Theorem 2.1.** (Picard-Lindelöf Theorem) Consider the initial value problem

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t)) \\
x(t_0) &= x_0.
\end{align*}
\]

Suppose \( f \) is uniformly Lipschitz continuous in \( x \) (Lipschitz constant can be taken independent of \( t \)) and continuous in \( t \). Then, for some \( \varepsilon > 0 \), there exists a unique solution \( x(t) \) to the initial value problem on the interval \( [t_0 - \varepsilon, t_0 + \varepsilon] \).

**Exercise 2.2.** Give an example of a function that is not Lipschitz and doesn’t have a unique solution.

*Answer:* Let \( f(x) = x^{\frac{1}{3}} \) and \( x(0) = 0 \). We know that \( x(t) = 0 \) is a solution to the ode \( \dot{x}(t) = f(x) \). Solving this ODE you get

\[
\begin{align*}
\frac{dx}{x^{\frac{1}{3}}} &= dt \\
\int_0^x \frac{dx}{x^{\frac{1}{3}}} &= \int_0^t dt \\
\frac{3}{2} x^{\frac{2}{3}} - 0 &= t \\
x(t) &= \frac{2}{3} t^{\frac{3}{2}} \text{ is a solution.}
\end{align*}
\]

We can see \( f(x) \) is not Lipschitz easily, the derivative at \( 0 \) becomes infinite.

**Exercise 2.3.** Give examples of Lipschitz functions with unique solutions.

*Answer:* Let \( f(x, t) = t^{\frac{1}{3}} \) and \( x(0) = 0 \). Then the only solution to \( \dot{x}(t) = f(x, t) \) is

\[
x(t) = \frac{3}{4} t^{\frac{4}{3}}.
\]

Another answer is \( f(x, t) = t^{\frac{1}{3}} x \), with \( x(0) = x_0 \). Solving the ODE \( \dot{x}(t) = f(x, t) \) using separation of variables gets us that

\[
x(t) = x_0 e^{\frac{3}{4} t^{\frac{3}{4}}}
\]

is the only solution.

**Exercise 2.4.** When do solutions exist for all time? What if they do not? How does this relate to the Kepler problem?

*Answer:* Solutions exist for all time as long as they stay inside a bounded region. Additionally, if a function is globally Lipschitz, then there exists a solution for all time. Another condition is that \( f \) having ”at most linear growth” implies global existence.

If a solution does not exist for all time, it is due to finite time blow-up. An example of an ODE that exhibits such behavior is

\[
\dot{x} = x^2,
\]

a separable ODE with the solution

\[
x(t) = \frac{1}{C - t}, \quad C \in \mathbb{R}.
\]

Drawing the solutions, you see you approach an asymptote as \( t \to C \) and so \( x(t) \) does not exist for all \( t \).
The Kepler problem has solutions that exist for all time, except when there are collisions. However, you can regularize collisions of two bodies, then solutions do exist for all time.

**Theorem 2.5.** (Local Implicit Function Theorem) Suppose $X, Y, Z$ are Banach spaces, $U \subset X$, $V \subset Y$ are open, $F : U \times V \to Z$ is continuously differentiable, $(x_0, y_0) \in U \times V$, $F(x_0, y_0) = 0$, and $D_x F(x_0, y_0)$ has a bounded inverse. Then there is a neighborhood $U_0 \times V_1 \subset U \times V$ of $(x_0, y_0)$ and a function $f : V_1 \to U_0$, $f(y_0) = x_0$ such that $F(x, y) = 0$ for $(x, y) \in U \times V$ \iff $x = f(y)$. If $F \in C^k(U \times V, Z)$, $k \geq 1$ or analytic in a neighborhood of $(x_0, y_0)$, then $f \in C^k(V_1, X)$ or is analytic in a neighborhood of $y_0$.

**Exercise 2.6.** If $\dot{x} = f(x)$ and $x(0) = x_0 \in \mathbb{R}^n$, with $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$, $k \geq 1$, what can we say about existence of a local flow?

*Answer:* Then, $\exists \delta > 0$, $U(x_0)$ such that $x(t, y_0), y_0 \in U(x_0)$, $|t| \leq \delta$, and $x(0, y_0) = y_0$ exists and is of class $C^k$ with respect to $t, y_0$. This means basically that for any initial condition, LOCALLY, I can solve for some time. In other words, we have a $C^k$ local (in time) flow!

**Exercise 2.7.** If the neighborhood in which $f$ is defined has no boundary and $x(t)$ never goes to infinity, what can we conclude about the time interval on which the flow is defined?

*Answer:* The maximal interval of existence of solutions is then all of $\mathbb{R}$.

**Lemma 2.8.** (Gronwall Lemma) If

$$x(t) \leq a(t) + \int_{t_0}^t b(s)x(s)\,ds$$

where $x, a, b \geq 0$ are continuous, then

$$x(t) \leq a(t) + \int_{t_0}^t a(s)b(s)e^{\int_s^t b(\sigma)\,d\sigma}\,ds$$
3. Stability

Consider the system

\[
\begin{align*}
\dot{x} &= f(x(t)) \\
x(0) &= x_0
\end{align*}
\]

where \(x(t) \in U \subseteq \mathbb{R}^n\), \(U\) an open set containing the origin and \(f : U \to \mathbb{R}^n\) continuous on \(U\). Suppose \(f\) has an equilibrium at \(x^*\), so \(f(x^*) = 0\).

**Definition 3.1.** The equilibrium \(x^*\) is said to be **Lyapunov stable** if for every \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(\|x(0) - x^*\| < \delta\), then for every \(t \geq 0\) we have \(\|x(t) - x^*\| < \varepsilon\).

**Remark.** Lyapunov stability of an equilibrium means that solutions starting “close” to the equilibrium will remain “close” forever.

**Definition 3.2.** The equilibrium \(x^*\) is said to be **asymptotically stable** if it is Lyapunov stable and there exists \(\delta > 0\) such that if \(\|x(0) - x^*\| < \delta\), then \(\lim_{t \to \infty} \|x(t) - x^*\| = 0\).

**Remark.** Asymptotic stability means that solutions that start “close” not only remain close (Lyapunov stability) but in fact will converge to the equilibrium point.

**Exercise 3.3.** Give an example of a system with an equilibrium point that is Lyapunov stable but is *not* asymptotically stable.

**Answer:** Consider the system

\[
\dot{u}(t) = 0.
\]

The solutions are constant, so all solutions are equilibria. However, none of these solutions are asymptotically stable. Another example is the system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -\sin(x)
\end{align*}
\]

This is the pendulum. The first integral, energy, is conserved for any solution and is given by

\[
I(x, y) = \frac{1}{2}y^2 - \cos(x).
\]

Drawing the picture, it’s easy to see why we have Lyapunov stability but not asymptotic stability (lots of orbits!).
4. Dynamics near Equilibria

**Theorem 4.1.** (Hartman-Grobman Theorem) Suppose $f$ is a differentiable vector field with
0 as a hyperbolic fixed point. Denote $\varphi_t(x)$ the corresponding flow and by $A = Df \bigg|_{x=0}$
the Jacobian matrix of $f$ evaluated at $x = 0$. Then, there is a homeomorphism

$$\psi(x) = x + h(x)$$

with $h$ bounded such that

$$\psi \circ e^{tA} = \varphi_t \circ \psi$$
in a sufficiently small neighborhood of 0. That is, there exists a neighborhood $U$ of the equi-
librium point 0 and a homeomorphism $\psi : U \rightarrow \mathbb{R}^n$, such that $\psi(0) = 0$ and such that in
the neighborhood $U$ the flow of $\dot{x}(t) = f(x)$ is topologically conjugate by the continuous map
$X = \psi(x)$ to the flow of its linearization $\dot{X} = AX$.

**Remark.** Hartman-Grobman is saying that the behavior of a nonlinear system is basically the
same as that of the linearized system as long as we stay away from the centers.

**Exercise 4.2.** Is the hyperbolicity condition necessary?

**Answer:** A hyperbolic equilibrium point or hyperbolic fixed point is a fixed point that does not
have any center manifolds. So, the Jacobian matrix at the equilibrium point does not have an
eigenvalue with real part 0. This condition is necessary, consider the system

$$\dot{x} = -x^3.$$  

This system has an equilibrium point at $x = 0$ and it is a sink. However, the linearized system
is given by

$$\dot{y} = 0,$$

which has a line of fixed points (centers).

**Exercise 4.3.** Is $\psi$ smooth?

**Answer:** $\psi$ is sometimes smooth (in particular when $f$ is), but the necessary and sufficient
conditions are not known. Sufficient conditions for $\psi \in C^k$ are non-resonance conditions.

4.1. **Stable Manifold Theorem.**

**Definition 4.4.** Let $X$ be a topological space and $f : X \rightarrow X$ a homeomorphism. Let $\phi_t(x)$
be the flow associated with the system $\dot{x} = f(x)$. If $x^*$ is a fixed point for $f$, the **stable set of
$x^*$** is defined by

$$W_s(x^*) = \{ x \in X \mid \lim_{t \rightarrow \infty} |\phi_t(x) - x^*| = 0 \}.$$ 

**Remark.** We define the unstable set, $W_u(x^*)$, similarly. Just use backward time! This follows
for everything in this section.

**Remark.** The stable and unstable sets are invariant under the flow.

**Definition 4.5.** Given a neighborhood $U$ of $x^*$, the **local stable manifold** of a fixed point
$x^*$ is

$$W^s_{loc}(x^*, U) = \{ x \in U \mid \gamma^+(x) \subset U \text{ and } \lim_{t \rightarrow \infty} \phi_t(x) = x^* \}$$

where $\gamma^+(x)$ represents the forward orbit of $x$.

**Remark.** On $W^s_{loc}$, solutions eventually leave any small neighborhood of the fixed point.

**Definition 4.6.** A set $S$ is **forward invariant** if $\phi_t(S) \subset S$ for all $t \geq 0$. 


Theorem 4.7. (Stable Manifold Theorem) Suppose \( \dot{x} = f(x) \) where \( f : X \to X \) has a fixed point \( x^* \) which is hyperbolic. Suppose as well that \( f \in C^k \), \( k \geq 1 \). Suppose as well that there exists a neighborhood \( U(x^*) \) a subset of the linearized system’s stable manifold, \( E^s(x^*) \). Then, there exists \( h \in C^k \) with \( h : U(x^*) \to E^s \) such that \( h(x^*) = 0 \), \( h'(x^*) = 0 \) and the local stable manifold

\[
W^s_{loc}(x^*) = \text{graph}(h) \subseteq X \subseteq \mathbb{R}^n
\]

is

(i) forward invariant,
(ii) ”maximal” in the sense that if \( \exists V(x^*) \subseteq X \) such that \( \varphi_t(x) \in V(x^*) \) for all \( t \geq 0 \), then \( \varphi_t(x) \in W^s_{loc}(x^*) \) for all \( t \geq 0 \)
(iii) for \( x_0 \in W^s_{loc}(x^*) \),

\[
|\varphi_t(x_0)| \leq C|x_0|e^{-\eta t}
\]

for some \( C, \eta > 0 \) independent of \( x_0 \).

Remark. The theorem states that \( W^s(x^*) \) is a smooth manifold and its tangent space has the same dimension as the stable space of the linearization of \( f \) at \( x^* \).

Exercise 4.8. What are the properties of the global stable manifold?

Answer:

(1) Invariant
(2) Consists of all solutions that converge to \( x^* \), the equilibrium point
(3) Unique manifold with property (2)
(4) \( W^s(x^*) \) is a manifold. You can think of \( W^s(x^*) \) as

\[
W^s_{loc}(x^*) = \bigcup_{t \leq 0} \varphi_t(W^s_{loc})
\]

and thus since \( \varphi_t(x^*) \) is a diffeomorphism and \( W^s_{loc}(x^*) \) is a manifold, \( \varphi_t(W^s_{loc}) \) is a diffeomorphic image of a manifold and hence a manifold.

(5) \( W^s(x^*) \) is not an embedded manifold! (i.e. \( x = x - x^3 \))

4.2. Center Manifold Theorem.

Definition 4.9. Let \( \dot{x} = f(x) \) be a dynamical system with equilibrium point \( x^* \). The linearization of the system near the equilibrium point is \( \dot{y} = Ay \) where \( A = Df(x^*) \). The Jacobian matrix \( A \) defines three main subspaces:

(1) The **stable subspace**, which is spanned by the generalized eigenvectors corresponding to the eigenvalues \( \lambda \) where \( \text{Re}(\lambda) < 0 \);
(2) The **unstable subspace**, which is spanned by the generalized eigenvectors corresponding to the eigenvalues \( \lambda \) where \( \text{Re}(\lambda) > 0 \);
(3) The **center subspace**, which is spanned by the generalized eigenvectors corresponding to the eigenvalues \( \lambda \) where \( \text{Re}(\lambda) = 0 \).

Definition 4.10. Corresponding to the linearized system, the nonlinear system has invariant manifolds, each consisting of sets of orbits of the nonlinear system.

(1) An invariant manifold tangent to the stable subspace and with the same dimension is the **stable manifold**;
(2) The **unstable manifold** is of the same dimension and tangent to the unstable subspace;
(3) A **center manifold** is of the same dimension and tangent to the center subspace. If, as is common, the eigenvalues of the center subspace are all precise zero, rather than just real part zero, then a center manifold is often called a **slow manifold**.
**Theorem 4.11.** (Center Manifold Theorem) If \( \dot{x} = Ax + g(x) \), where \( g \in C^k \) is \( O(x) \) with \( k < \infty \), then there exists \( h : U(x^s) \subseteq E^c \to E^s \oplus E^u \) with
\[
\text{graph}(h) = W^c \in C^k,
\]
which is locally invariant. Additionally, there exists \( \varepsilon > 0 \) such that \( W^c \) contains all solutions with
\[
|x(t)| < \varepsilon
\]
for all \( t \).
5. Floquet Theory

Theorem 5.1. (Floquet’s Theorem) If $\dot{y} = A(t)y$, where $A(t)$ is continuous, $T$-periodic, and $A(t) \in \mathbb{C}^{n \times n}$, then there exists $B, P(t) \in \mathbb{C}^{n \times n}$ with

1. $P$ invertible
2. $P(t) = P(t + T)$, that is, $P$ is $T$-periodic
3. $P(0) = \text{id}$

so that $\varphi_{t,0} = P(t)e^{Bt}$.

Exercise 5.2. What is $\varphi_{t,0}$?
Answer: It is the fundamental solution to

$\dot{y} = A(t)y$

where $A(t)$ is $T$-periodic, with the properties

1. $y(t) = \varphi_{t,s}y(s)$,
2. $\varphi_{t,\tau}\varphi_{\tau,s} = \varphi_{t,s}$,
3. $\varphi_{t+T,s+T} = \varphi_{t,s}$.

Exercise 5.3. What are Floquet exponents?
Answer: The elements of $\text{spec}(B)$; i.e. the eigenvalues of $B$ since we’re in finite dimensions.

Exercise 5.4. What are Floquet multipliers?
Answer: The elements of $\text{spec}(e^{BT})$, i.e. the eigenvalues of $\varphi_{T,0} = e^{Bt}$.
6. **Hamiltonian Systems**

Consider a sufficiently differentiable function \( H(q,p,t) \) where \( q,p \in \mathbb{R}^n \) and \( H : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R} \). A Hamiltonian system is a system of \( 2n \) ODEs of the form

\[
\begin{align*}
\dot{q} &= H_p \\
\dot{p} &= -H_q
\end{align*}
\]

where \( H \) is called the Hamiltonian. The vectors \( q \) and \( p \) are traditionally referred to as position and momentum, respectively. It can be written in the compact form

\[
\dot{z} = J \nabla H(z,t), \quad z = (q,p),
\]

where \( J \) is the canonical symplectic matrix

\[
J = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}.
\]

Basic existence and uniqueness tells us that there is a solution \( z = \varphi_t(z) \) of \( 3 \) satisfying a given initial condition \( z(t_0) = z_0 \).

**Note.** Consider the special case when \( H \) is a function of only \( q \) and \( p \). The differential equations \( 3 \) are then autonomous and the Hamiltonian system is called conservative.

**Definition 6.1.** An integral of motion for \( 3 \) is a smooth function \( F \) which is constant along the solutions of \( 3 \). The classical conserved quantities of energy, momentum, etc. are integrals.

**Exercise 6.2.** Give an example of a Hamiltonian system and an integral of motion.

**Answer:** The central force problem has the system of equations

\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= k(|q|)q
\end{align*}
\]

where the Hamiltonian is given by

\[
H(q,p) = \frac{1}{2}|p|^2 - K(|q|)
\]

where \( K(|q|) \) is the integral of \( k(|q|) \). The angular momentum, \( c = q \times p \) is a vector of integrals since

\[
\dot{c} = q \times \dot{p} + \dot{q} \times p = q \times k(|q|)q + p \times p = 0.
\]

\( H(q,p) \) itself is an integral of motion.

**Theorem 6.3.** If \( \lambda \) is an eigenvalue of a linearized Hamiltonian system, then so are \( -\lambda, \lambda, \) and \( -\lambda \).

**Remark.** Hamiltonian systems cannot be linearly stable (all eigenvalues of negative real part).

**Remark.** The above theorem follows from the fact that the characteristic polynomial of a real Hamiltonian matrix is an even polynomial. So the characteristic polynomial for \( \lambda \) the eigenvalue \( \rho(\lambda) = \det(A - \lambda I) = \det(A + \lambda I) = \rho(-\lambda) \).

6.1. **Volume Preserving Flow.**

**Theorem 6.4.** (Liouville’s Theorem) The volume in phase space is preserved under a Hamiltonian flow.
**Proof.** First, it is important to note that Hamiltonian systems are divergence free. That is, if we let $F$ represent the Hamiltonian vector field, $\nabla \cdot F = 0$. This follows from Clairaut’s Theorem. We know from the change of variables theorem that if we have a map $G : \mathbb{R}^n \to \mathbb{R}^n$ and set $V \subset \mathbb{R}^n$, then

$$\text{vol}(V) = \int_V d\mathbf{x} \quad \text{and} \quad \text{vol}(G(V)) = \int_V |\det(M)| \, d\mathbf{x},$$

where $M = DG$, the Jacobian. So, we can say that $G$ is a volume-preserving map if $|\det(M)| = 1$. Let us rewrite 3 as

$$\dot{z} = f(z, t).$$

We want to look at the determinant of the Jacobian of this flow. So, let $M = D\varphi_t(z)$. Let $\varphi_t(z)$ represent the flow map of this system. Then,

$$\frac{d}{dt}D\varphi_t(z) = f(\varphi_t(z)),$$

where both sides are vector valued functions of $z$ and $t$. Then, computing the Jacobian of both sides and changing the order of integration, we can see that

$$\frac{d}{dt}D\varphi_t(z) = Df(\varphi_t(z))D\varphi_t(z) \implies \frac{dM}{dt} = Df(\varphi_t(z)) = M.$$

Assuming $M$ is invertible ($\det(M) \neq 0$), we can multiply both sides on the right by $M^{-1}$ and we get

$$\dot{MM}^{-1} = Df(\varphi_t(z)).$$

Take the trace of both sides and you get

$$\text{tr}(\dot{MM}^{-1}) = \text{tr}(Df(\varphi_t(z))).$$

First, notice that $\text{tr}(F') = \nabla \cdot F$. In this case, that means that

$$\text{tr}(Df(\varphi_t(z))) = \nabla \cdot Df(\varphi_t(z)) = 0.$$

From Jacobi’s formula for the derivative of a determinant, we get

$$\text{tr}(\dot{MM}^{-1}) = \frac{d}{dt} \frac{\det(M)}{\det(M)}.$$

Thus, we get

$$\frac{d}{dt} \frac{\det(M)}{\det(M)} = 0.$$

Since $M(0) = I$ and $\frac{d}{dt} \det(M) = 0$, we know that $\det(M) = 1$. \hfill \Box

**Remark.** Hamiltonian systems do not have attractors as a direct consequence of Liouville’s theorem (asymptotic stability). Suppose that a Hamiltonian system did have an attractor. Then, there exists a set around the attractor whose points tend to the attractor asymptotically. So the image of this set would contract to the attractor under the flow and so since the set contains the attractor, the volume of the image of the set decreases.

**Remark.** In fact, Liouville’s theorem holds for any divergence free vector field.

**Exercise 6.5.** Under what conditions is a planar ODE guaranteed a first integral?

**Answer:** If

$$\begin{cases}
    \dot{x} = f(x, y) \\
    \dot{y} = g(x, y)
\end{cases}$$

and $\partial_x f + \partial_y g = 0$ (divergence of vector field is 0) for all values of $x$ and $y$, then there exists a first integral $I = \partial_y I$ and $g = -\partial_x I$ such that

$$\frac{dI}{dt} = 0.$$
I exists since $\partial_{xy}I = \partial_x f = \partial_y f = -\partial_y g$ (if simply connected of course). I is called the Hamiltonian.

6.2. Symplectic Structure.

**Definition 6.6.** A matrix $M$ is **symplectic** if $MJM^T = J$.

**Definition 6.7.** If the Jacobian of a transformation $(q, p) \rightarrow (Q, P)$ is symplectic, then it is a **canonical transformation**.

**Remark.** A canonical transformation will preserve the Hamiltonian form of equations.

**Theorem 6.8.** (Poincaré, 1899) Let $H(q, p)$ be a twice continuously differentiable function on $U \subset \mathbb{R}^{2n}$. Then, for each fixed $t$, the flow $\varphi_t$ is a symplectic transformation wherever it is defined.
7. Celestial Mechanics

Celestial mechanics is the study of point particles in $\mathbb{R}^3$ moving under the influence of their mutual gravitational attraction. A point particle is characterized by a position $q = (x, y, z) \in \mathbb{R}^3$, and a mass $m \in \mathbb{R}^+$. The motion of a particle is a smooth curve $q(t), t \in \mathbb{R}$ representing time. Then, we can define the velocity $v(t) = \dot{q}(t) \in \mathbb{R}^3$ and momentum $p(t) = mv(t) \in \mathbb{R}^3$.

Newton’s equation of motion gives us that

$$F(q, t) = ma(t) = m\ddot{q}(t),$$

where $F(q, t)$ is the force acting on the particle.

Now, consider $n$ point particles with positions $q_i = (x_i, y_i, z_i)$ and corresponding masses $m_i$. So, we can introduce new notation $q = (q_1, \cdots, q_n) \in \mathbb{R}^{3n}$ and so you can write the velocity $v(t)$ and momentum $p(t)$ as

$$v = \dot{q},$$
$$p = Mv$$

where $M$ is the $3n \times 3n$ matrix

$$M = \begin{pmatrix}
  m_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
  0 & m_1 & 0 & \cdots & \cdots & \cdots & 0 \\
  0 & 0 & m_1 & \cdots & \cdots & \cdots & 0 \\
  0 & \cdots & 0 & m_2 & \cdots & \cdots & 0 \\
  0 & \cdots & \cdots & 0 & \ddots & \cdots & \cdots & \cdots & \cdots & m_n & 0 & 0 \\
  0 & \cdots & \cdots & \cdots & 0 & m_n & 0 & 0 \\
  0 & \cdots & \cdots & \cdots & \cdots & 0 & m_n & \cdots & \cdots & \cdots & 0 & m_n
\end{pmatrix}.$$  

The gravitational force acting on particle $i$ due to particle $j$ is known to be (Newton)

$$F_{ij} = \frac{m_im_j(q_j - q_i)}{|q_j - q_i|^3}$$

and so the total force on particle $i$ is the sum of all the forces acting on it:

$$F_i = \sum_{j \neq i} F_{ij}.$$  

Another way to write this is as a gradient of the Newtonian potential $U(q)$

$$F_i = \nabla_i U(q)$$

with

$$U(q) = \sum_{(i,j), i < j} \frac{m_im_j}{|q_i - q_j|} \quad \text{and} \quad \nabla_i = \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i} \right).$$

Remark. The Newtonian interparticle potential has two important properties.

(i) The $m_im_j$ factor causes the mass $m_i$ to cancel out of the equation of motion of $q_i$, which agrees with Galileo’s observation that the behavior of a falling body is independent of its mass.

(ii) The function $\frac{1}{|q_i - q_j|}$ is a harmonic function of $q_i \in \mathbb{R}^3$. It follows (Gauss) that the potential due to a spherically symmetric mass distribution is the same as if the whole mass were concentrated at the center. Thus, we can use point particles.
The equations of the Newtonian \( n \)-body problem are
\[
\dot{\mathbf{q}} = M\ddot{\mathbf{q}} = \nabla U(\mathbf{q})
\]
as \( i = 1, \ldots, n \), which follows from the definition of momentum and Newton’s equation of motion. Putting the whole system together we get
\[
\dot{\mathbf{p}} = M\ddot{\mathbf{q}} = \nabla U(\mathbf{q}).
\]
So, we have a real analytic system of second order ODEs on the configuration space
\[
X = \mathbb{R}^{3n} \setminus \Delta,
\]
where \( \Delta = \{ q \mid q_i = q_j \text{ for some } i \neq j \} \), the collision set.

We rewrite this as the system of first order equations
\[
\begin{cases}
\dot{\mathbf{q}} = \mathbf{v} \\
\dot{\mathbf{v}} = M^{-1} \nabla U(\mathbf{q})
\end{cases}
\]
on the phase space \( TX = \{ (q, v) \mid q \in X, v \in \mathbb{R}^{3n} \} \subset \mathbb{R}^{6n} \), where \( TX \) is the tangent bundle of \( X \).

7.1. Hamiltonian System. If instead of \( q \) and \( v \) we use \( q \) and \( p \), we can find that we have a Hamiltonian system. The Newtonian \( n \)-body problem is a Hamiltonian system with Hamiltonian
\[
H : T^*X \to \mathbb{R}
\]
\[
H(q, p) = \frac{1}{2} pM^{-1} p^T - U(q),
\]
where \( T^*X \) is the cotangent bundle of \( X \). Thus, we get Hamilton’s equations
\[
\begin{cases}
\dot{q} = \frac{\partial H}{\partial p}(q, p) \\
\dot{p} = -\frac{\partial H}{\partial q}(q, p)
\end{cases}
\]
(4)
\( q \) and \( p \) are conjugate variables.

7.2. Change of Coordinates. Consider new coordinates \( (Q, P) \) related to the old ones by the diffeomorphism
\[
\begin{cases}
q = q(Q, P) \\
p = p(Q, P)
\end{cases}
\]
Let
\[
K(Q, P) = H(q(Q, P), p(Q, P))
\]
and suppose that
\[
p(Q(t), P(t))\dot{q}(Q(t), P(t)) = P(t)\dot{Q}(t)
\]
for every curve \( (Q(t), P(t)) \). This is the same as requiring the equality of differential forms
\[
p \, dq = P \, dQ.
\]
It follows then that Hamilton’s equations hold in the new coordinates
\[
\begin{cases}
\dot{Q} = \frac{\partial K}{\partial P} \\
\dot{P} = -\frac{\partial K}{\partial Q}
\end{cases}
\]

Exercise 7.1. Consider Jacobi coordinates (rotating coordinates) for the 3-body problem. First, define
\[
M = m_1 + m_2 + m_3
\]
and the center of mass
\[
\bar{q} = \frac{1}{M}(m_1 q_1 + m_2 q_2 + m_3 q_3).
\]
Define new variables \( Q_1 = q_2 - q_1 \) and \( Q_2 = q_3 - (\gamma_1 q_1 + \gamma_2 q_2) \) where \( \gamma_1 := \frac{m_1}{m_1 + m_2} \) and \( \gamma_2 := \frac{m_2}{m_1 + m_2} \). So, \( Q_1 \) represents the distance between \( q_1 \) and \( q_2 \) while \( Q_2 \) represents the distance
between \( q_3 \) and halfway between \( q_1 \) and \( q_2 \). Now, find the conjugate momenta (mass times derivative of position):
\[
\overline{p} = \frac{1}{M} \dot{\overline{q}}, \quad P_1 = \alpha \dot{Q}_1, \quad P_2 = \beta \dot{Q}_2.
\]

where \( \alpha := \frac{m_1m_2}{m_1 + m_2} \) and \( \beta := \frac{(m_1 + m_2)m_3}{M} \).

It can be shown that
\[
\overline{p}d\overline{q} + P_1dQ_1 + P_2dQ_2 = p_1dq_1 + p_2dq_2 + p_3dq_3.
\]

Thus, the Hamiltonian structure still holds and so we can simply find our new Hamiltonian to find the new equations. Our new Hamiltonian is given by
\[
K(Q, P) = \frac{1}{2M} |\overline{p}|^2 + \frac{1}{2\alpha} |P_1|^2 + \frac{1}{2\beta} |P_2|^2 - U(Q)
\]

where \( U(Q) = \frac{m_1m_2}{|Q_1|} + \frac{m_1m_3}{|Q_2 + \gamma Q_1|} + \frac{m_2m_3}{|Q_2 - \gamma Q_1|} \)

7.3. Symmetries and Integrals. The Hamiltonian of the \( n \)-body problem is symmetric under the action of the Euclidean group Euc(3). An element \( g \in \text{Euc}(3) \) takes the form
\[
g \cdot x = Ax + b
\]

where \( A \) is a \( 3 \times 3 \) orthogonal matrix \( (A^T = A^{-1} \) and \( \det(A) = \pm 1) \), \( b \in \mathbb{R}^3 \), and \( x \in \mathbb{R}^3 \). If we want to extend this action onto elements of the form \((q, p)\), we get that it acts as follows:
\[
g \cdot (q, p) = g \cdot \left( \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \right) = \left( \begin{pmatrix} g \cdot q_1 \\ \vdots \\ g \cdot q_n \end{pmatrix}, \begin{pmatrix} g_1^T p_1 + \cdots + g_n^T p_n \end{pmatrix} \right)
\]

Then, we can show that since \( A \) is orthogonal,
\[
\begin{align*}
H(g \cdot (q, p)) &= \frac{1}{2} \sum_i m_i^{-1} (p_i A^T) (p_i A^T)^T - \sum_{i < j} \frac{m_i m_j}{|g \cdot q_i - g \cdot q_j|} \\
&= \frac{1}{2} \sum_i m_i^{-1} p_i A^T p_i - \sum_{i < j} \frac{m_i m_j}{|A(q_i - q_j)|} = H(q, p)
\end{align*}
\]

So it follows that \((q(t), p(t))\) solves the \( n \)-body problem iff \( g \cdot (q(t), p(t)) \) solves the \( n \)-body problem.

Let \( g_s \) be a one-parameter family of Euclidean transformations where \( g_0 = I \). Define the vector field
\[
X(q) = \frac{d}{ds} g_s(q) \bigg|_{s=0}.
\]

Noether’s Theorem tells us that if \( g_s \) is a family of symmetries of the Hamiltonian \( H(q, p) \), then \( F(q, p) = pX(q) \) is an integral of Hamilton’s equations. That is to say,
\[
F(q(t), p(t)) = C, \quad C \in \mathbb{R}
\]

if \((q(t), p(t))\) is a solution. Using this, we can derive the classical integrals of the \( n \)-body problem. Let
\[
g_s \cdot x = x + sb, \quad b \in \mathbb{R}^3.
\]

Then,
\[
g_s \cdot q = \begin{pmatrix} q_1 + sb \\ \vdots \\ q_n + sb \end{pmatrix}
\]
so we get that the vector field is
\[ X(q) = \begin{pmatrix} b \\ \vdots \\ b \end{pmatrix} \]
and
\[ F(q,p) = pX(q) = (p_1 \cdots p_n) \begin{pmatrix} b \\ \vdots \\ b \end{pmatrix} = \left( \sum_{i=1}^n p_i \right) b. \]
Since this is a constant for all \( b \in \mathbb{R}^3 \), the quantity
\[ p = \sum_{i=1}^n p_i \]
is a form-valued integral called the total momentum. The value of \( p \) determines the motion of the center of mass
\[ q = \frac{1}{M} \sum_{i=1}^n m_i q_i, \quad M = \sum_{i=1}^n m_i. \]
Clearly,
\[ \dot{q} = \frac{1}{M} \dot{p} \]
and so \( \overline{q}(t) = \overline{q}(t_0) + \frac{1}{M} \overline{p}(t - t_0) \). If \((q(t), p(t))\) is any solution of the \( n \)-body problem with momentum \( \overline{p} \), then
\[ \dot{q}_i(t) = q_i(t) - q(t) \text{ and } \dot{p}_i(t) = p_i(t) - p(t) \]
is also a solution, but with total momentum 0. Thus, we can study solutions with total momentum 0 without loss of generality. In fact, when the total momentum is 0, \( \overline{q} \) becomes a vector valued constant of motion, and the solution \( \tilde{q}(t) \) also has center of mass equal to 0, so we can in fact assume the center of mass is at the origin.

We can find another integral by applying Noether’s theorem to a one-parameter family of rotations with constant angular velocity \( \omega \in \mathbb{R}^3 \), instead of translations. This leads to the conservation of angular momentum, \( \Omega = \sum_{i=1}^n q_i \times p_i^T \in \mathbb{R}^3 \).

The integrals \( \overline{p}, \overline{q}, \) and \( \Omega \) provide 9 constants of motion (at least if \( \overline{p} = 0 \)). The Hamiltonian
\[ H(q,p) = \frac{1}{2} pM^{-1} p^T - U(q) \]
is the 10th.

There is one more symmetry we can invoke.
\[ S_{(\omega_0), h} = \{(q,p) \in T^*X \mid \overline{q} = 0, \overline{p} = 0, \Omega = \omega_0, H = h\} \]
is still invariant under orthogonal transformations which fix \( \omega_0 \in \mathbb{R}^3 \). In particular, there is a one parameter rotational symmetry. This means that there is a well-defined dynamical system on the quotient space \( S'_{(\omega_0), h} \), which will be a manifold of dimension \((6n - 11)\) if \( S_{(\omega_0), h} \) is a manifold of dimension \((6n - 10)\).

7.4. Rescalings. If \((q(t), p(t))\) is a solution in \( S_{(\omega_0), h} \), then for any constant \( \sigma \neq 0 \),
\[ \tilde{q}(t) = \sigma^{-2} q(\sigma^3 t), \quad \tilde{p}(t) = \sigma p(\sigma^3 t) \]
is a solution in \( S'_{(\sigma^{-1} \omega_0), \sigma^2 h} \). Thus, there is only really one parameter in the \( n \)-body problem (besides the masses), \(|\omega_0|^2 h\). By means of this scaling, we can normalize the energy \( h \) to \(-1, 0, \) or 1 as necessary.
We can also rescale to normalize the masses. If \((q(t), p(t))\) is a solution to the \(n\)-body problem for masses \(m_1, \ldots, m_n\), then for any constant \(\mu \neq 0\),
\[
\dot{q}(t) = q(\mu t), \quad \dot{p}(t) = \mu^3 p(\mu t)
\]
is a solution for masses \(\mu^2 m_1, \ldots, \mu^2 m_n\). It is common to normalize the masses so that
\[
M = \sum_{i=1}^{n} m_i = 1.
\]

7.5. The Two-Body Problem. When \(n = 2\), \(6n - 11 = 1\) and so we have a manifold of dimension 1 for our problem. This completely solves it. Our solutions live on the space
\[
X = \{ (q_1, q_2) \in \mathbb{R}^6 \mid q_1 \neq q_2 \}, \quad T^*X = \{ (q_1, q_2, p_1, p_2) \} \subset \mathbb{R}^7 \times \mathbb{R}^{6*}.
\]
our equations of motion are given to be
\[
\begin{aligned}
\dot{q}_1 &= m_1^{-1} p_1 \\
\dot{p}_1 &= \frac{m_1 m_2 (q_2 - q_1)}{|q_2 - q_1|^3} \\
\dot{q}_2 &= m_2^{-1} p_2 \\
\dot{p}_2 &= \frac{m_1 m_2 (q_1 - q_2)}{|q_1 - q_2|^3}
\end{aligned}
\]
Now, we scale the masses so they sum to 1 and introduce a new coordinate which is the distance between the two bodies.
\[
\text{center of mass: } \overline{q} = m_1 q_1 + m_2 q_2 \\
\text{total momentum: } \overline{p} = p_1 + p_2 \\
\text{relative position: } Q = q_2 - q_1 \\
\text{relative velocity: } P = \frac{p_2}{m_2} - \frac{p_1}{m_1}
\]
Now we derive our new set of differential equations and we get
\[
\begin{aligned}
\dot{\overline{q}} &= \overline{p} \\
\dot{\overline{p}} &= 0 \\
\dot{Q} &= P \\
\dot{P} &= \frac{-Q}{|Q|^3}
\end{aligned}
\]
Elimination of the total momentum and the center of mass are accomplished by ignoring \(\overline{q}\) and \(\overline{p}\). This leaves a Hamiltonian system on \(T^*(\mathbb{R}^3 \setminus \{0\})\).
\[
H(Q, P) = \frac{1}{2} |P|^2 - \frac{1}{|Q|}.
\]
This is called the Kepler problem. Assuming that \(\overline{p} = 0\), the angular momentum is given by
\[
\Omega(Q, P) = m_1 m_2 Q \times P.
\]