Reference Sheet

We are going to assume there is a given abstract set \( P \) of “pictures”.

**Definition 1.** Let \( f : \mathbb{N} \to \mathbb{N} \). Then we say \( F(x) \) is the ordinary generating function (ogf) for \( f \) if

\[
F(x) = \sum_{n \geq 0} f(n) x^n
\]

where the power series on the right hand side is understood to be a formal power series. That is, \( x \) is treated as an indeterminate variable. Similarly, we say \( G(x) \) is the exponential generating function (egf) for \( f \) if

\[
G(x) = \sum_{n \geq 0} \frac{f(n)}{n!} x^n
\]

where the power series is again understood to be formal.

**Definition 2.** A connected structure is a pair \( C = (S, p) \) consisting of a finite set \( S \) of positive integers (the “label” set) and a picture \( p \) using all labels from the labeling set. The weight of \( C \) is \( n = |S| \). A connected structure of weight \( n \) is called standard if its label set is \([n]\). For all practical purposes, our connected structures will always be standard and we view \( p \) as a permutation on \( n \) letters with a single cycle.

**Definition 3.** A weight class \( W \) is a set of standard connected structures whose weights are all the same and whose pictures are all different. The weight of \( W \) is the common weight of all the connected structures. For all practical purposes, \( W \) will be the set of connected structures of weight \( n \) whose pictures are all possible single cycle permutations on \( n \) letters.

**Example 1.** Let \( S = [3] \) and our structures be permutations. Then the connected structures on this labeling set are as follows:

\[
C_1 = ([3], (123)) \quad C_2 = ([3], (132))
\]

Notice how the set of connected structures is the set of elements of \( S_3 \) with a single cycle and \( W_3 = \{C_1, C_2\} \).

**Definition 4.** A disconnected structure \( D \) is a set of connected structures whose label sets form a partition of \([n]\), for some \( n \). The weight of \( D \) is the sum of the weights of the connected structures in \( D \). For all practical purposes, if the weight of \( D \) is \( n \), then we view \( D \) as any permutation on \( n \) letters.

**Example 2.** Consider the same setup as in Example 1. The disconnected structures on this set are as follows:

\[
D_1 = ([3], (123)) \quad D_2 = ([3], (132)) \quad D_3 = ([3], (1)(23)) \quad D_4 = ([3], (2)(13)) \quad D_5 = ([3], (3)(12)) \quad D_6 = ([3], (1)(2)(3))
\]

Notice \( D_1 = C_1, D_2 = C_2 \), and that the set \( \{D_i \mid 1 \leq i \leq 6\} \) is in bijective correspondence with \( S_3 \). This holds generally for when \( S = [n] \).
Definition 5. A family $\mathcal{F}$ is a collection of weight classes $W_1, W_2, \ldots$ where for each $n = 1, 2, \ldots$, the weight class $W_n$ is of weight $n$. Write $w_n$ for the number of connected structures in $W_n$, and we call $W(x)$, the egf of the sequence $\{w_n\}_{n \geq 0}$, the weight enumerator of the family. For all practical purposes, the weight classes of $\mathcal{F}$ are those of permutations.

Theorem 1 (The Exponential Formula). Let $\mathcal{F}$ be a family, $W(x)$ be the weight enumerator of $\mathcal{F}$, and $G(x)$ be the egf for the number of disconnected structures which can be built from $\mathcal{F}$. Then

$$G(x) = e^{W(x)}$$

Proof. For an induction based version of the proof, see Wilf’s Generatingfunctionology. \qed