Linear Independence and Dependence*

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Let us say we have a group of vectors and you want to understand how they are related to one another. For example, take the following set of vectors,

\[
\begin{align*}
  w_1 &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \\
  w_2 &= \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \\
  w_3 &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
  v_4 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\end{align*}
\]

What are some things we may notice about these vectors?

1. \(2w_1 = w_2\)
2. \(0w_1 = 0w_2 = 0w_4 = w_3\)
3. \(w_4 + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = w_1,\)

We say that vectors are linearly dependent if one is a scalar multiple of the other, like \(w_1\) and \(w_2\) above. Otherwise, we say that they are linearly independent, such as \(w_1\) and \(w_4\). More formally, we get the following definition.

**Definition (Linearly Dependent).** The set of vectors \(\{v_1, v_2, \ldots, v_n\}\) is linearly dependent if for some \(v_k\) in \(\{v_1, \ldots, v_n\}\) there exists constants \(c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n\) such that

\[
c_1v_1 + c_2v_2 + \ldots + c_{k-1}v_{k-1} + c_{k+1}v_{k+1} + \ldots + c_nv_n = v_k,
\]

that is, there exists a linear combination of the other vectors to get you \(v_k\).

It is important to note that not all the \(c_i\) need to be nonzero. Let us return to our example above.

We can easily see that

\[
2w_1 + w_3 + 0w_4 = w_2,
\]

but these constants are not unique. For example, \(c_3\) could really be any number. As long as \(c_1 = 2\) and \(c_4 = 0\), we’re fine.

So, a collection of vectors is linearly independent if we can’t do this - that is, we can’t find non-zero constants that such a relationship holds. More formally,

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Definition (Linearly Independent). The set of vectors \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent if the equation
\[
c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0
\]
can only be satisfied by \( c_i = 0 \) for \( i = 1, \ldots, n \). That is to say, no vector in the set can be represented as a linear combination of the remaining vectors in the set.

For example, the set of vectors
\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]
are linearly independent.

Now, here are some important things to notice.

(1) This should ring some bells about Gaussian elimination! If you made a matrix filled with rows equal to your vectors, you should be able to row reduce that matrix using Gauss-Jordan elimination. If you can make one row all zeroes, then your set is linearly dependent (since one row is a multiple of the other). Consider the vectors \( w_1, w_2, \) and \( w_3 \) from our first example. We know that \( w_2 = 2w_1 \), so let us construct a matrix and do Gaussian elimination.
\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
1 & 1 & 1
\end{pmatrix}
\]
\[
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]
and so, since we have a row of zeroes, we have linear dependence. We notice that saying \( 2w_1 = w_2 \) is the same as saying \( w_2 - 2w_1 = 0 \). This relationship is obvious, but sometimes the relationship is not obvious. Take for example the following vectors
\[
u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}.
\]

How can we tell if they are linearly independent or dependent? We can put them into a matrix and use Gaussian elimination.
\[
\begin{pmatrix}
1 & 2 & 3 \\
1 & -1 & 1 \\
3 & 0 & 5
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
0 & -3 & -2 \\
0 & -6 & -4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & 3 \\
0 & -3 & -2 \\
0 & 0 & 0
\end{pmatrix}
\]
We found a row of zeroes so this tells us we must have linear dependence. Great! But how can we use this to find the relationship between the vectors? From our row reduction steps, we learned that
\[
u_3 - 3u_1 = \begin{pmatrix} 0 \\ -6 \\ -4 \end{pmatrix}, \quad u_2 - u_1 = \begin{pmatrix} 0 \\ -3 \\ -2 \end{pmatrix}, \quad 2(u_2 - u_1) = \begin{pmatrix} 0 \\ -6 \\ -4 \end{pmatrix},
\]
where we see that the first and last two vectors are the same! Thus,
\[
u_3 - 3u_1 = 2(u_2 - u_1)
\]
\[
\Rightarrow u_3 = 2u_2 + u_1
\]
This is our linear dependence relationship, it tells us how to get one vector in terms of the others.
(2) If you have a set of linearly dependent vectors, then you can remove the vector(s) that have a dependence and not change the possible things that the other vectors sum to. So, if the set of vectors \{v_1, \ldots, v_n\} is fixed, then we might say that the span of these vectors is given by all possible linear combinations of \{v_1, \ldots, v_n\}. Thinking about it like this, then if \(v_n\) is equal to some linear combination of \(v_1, \ldots, v_{n-1}\), then
\[
\text{span} \{v_1, \ldots, v_n\} = \text{span} \{v_1, \ldots, v_{n-1}\}
\]
since we could just make \(v_n\) from \(v_1, \ldots, v_{n-1}\).

Consider once more the set \(u_1, u_2, u_3\) from the example above. Now, we saw that \(u_3 = u_1 + 2u_2\). So if there was some other vector, \(\tilde{v}\) that was equal to some linear combination of \(u_1, u_2, \text{ and } u_3\), we could write it without using \(u_3\). For example, say we knew that \(\tilde{v} = 6u_1 - 9u_2 - u_3\), we use our linear dependence relationship and we re-write this sum as
\[
\tilde{v} = 6u_1 - 9u_2 - (u_1 + 2u_2)
= 6u_1 - 9u_2 - u_1 - 2u_2
= 5u_1 - 11u_2
\]
and we see that we rewrite this without the linearly dependent vector.