STABILITY OF THE COHOMOLOGY OF THE SPACE OF COMPLEX
IRREDUCIBLE POLYNOMIALS IN SEVERAL VARIABLES

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Abstract. We prove that the space of complex irreducible polynomials of degree \( d \) in \( n \) variables satisfies two forms of homological stability: first, its cohomology stabilizes as \( d \to \infty \), and second, its compactly supported cohomology stabilizes as \( n \to \infty \). Our topological results are inspired by counting results over finite fields due to Carlitz and Hyde.

1. Introduction

The interests of counting irreducible polynomials over finite fields have stretched from eighteenth century to the modern era. Let \( \mathrm{Irr}_{d,n}(\mathbb{F}_q) \) denote the number of irreducible polynomials of total degree \( d \) in \( n \) variables with coefficients in \( \mathbb{F}_q \) up to scalar multiplications. For example, Gauss ([5], page 611) first calculated the size of \( \mathrm{Irr}_{d,1}(\mathbb{F}_q) \). In 1963, Carlitz [1] proved that for integers \( n>1 \)

\[
\frac{|\mathrm{Irr}_{d,n}(\mathbb{F}_q)|}{q^{\left(d+n-1\right)/2}} \to 1 + q^{-1} + q^{-2} + \cdots \quad \text{as } d \to \infty.
\]

(1.1)

In 2018, Hyde (Theorem 1.1 in [6]) proved that \( |\mathrm{Irr}_{d,n}(\mathbb{F}_q)| \) is always a polynomial in \( q \) which converges coefficient-wise to a formal power series \( P_d(q) \) as \( n \to \infty \). In other words, in the formal power series ring \( \mathbb{Q}[q] \) equipped with the \( q \)-adic topology (under which higher powers of \( q \) are considered smaller),

\[
|\mathrm{Irr}_{d,n}(\mathbb{F}_q)| \to P_d(q) \quad \text{as } n \to \infty.
\]

(1.2)

In this paper, we will pass from \( \mathbb{F}_q \) to \( \mathbb{C} \) and study the topology of the following manifold:

\[
\mathrm{Irr}_{d,n}(\mathbb{C}) := \{ \text{irreducible complex polynomials in } n \text{ variables with degree } d \}/\mathbb{C}^{*}.
\]

In [2], Church-Ellenberg-Farb used the Grothendieck-Lefschetz trace formula to connect asymptotic point-counts over finite fields and stability phenomena in cohomology. Heuristics based on this connection lead us to ask the following topological questions inspired by the aforementioned counting results of Carlitz and Hyde (see Section 2 for a brief explanation of the heuristics):

Question 1. Does \( H^i(\mathrm{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}) \) stabilize as \( d \to \infty \)?

Question 2. Does \( H^i_p(\mathrm{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}) \) stabilize as \( n \to \infty \)?

Observe that \( H^i_p(\mathrm{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}) \) is Poincaré dual to \( H^{D-i}(\mathrm{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}) \) where \( D \) is the real dimension of the manifold \( \mathrm{Irr}_{d,n}(\mathbb{C}) \). Thus, Question 2 equivalently asks if \( \mathrm{Irr}_{d,n}(\mathbb{C}) \) satisfies cohomological stability in codimensions.

We will prove the following two theorems, each respectively answering the questions above affirmatively.

Theorem 1.1. For \( n>1 \) and \( d \) any positive integer, when \( i \leq 2 \left(\binom{d+n-1}{n-1} - n - 1\right) \), we have

\[
H_i(\mathrm{Irr}_{d,n}(\mathbb{C}), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd}. \end{cases}
\]
Remark 1.1. Theorem 1.1 implies that $H^i(\text{Irr}_{d,n}(\mathbb{C}), \mathbb{Z})$ stabilizes as either $n$ or $d$ increases, giving a positive answer to Question 1. In fact, Carlitz also proved that the same limit in (1.1) holds as $n \to \infty$ (see equation (11) in [1]), although he didn’t state it in the main theorem. Thus, Theorem 1.1 can be viewed as a topological analog of Carlitz’ result.

Observe that there is a natural inclusion $\text{Irr}_{d,n}(\mathbb{C}) \hookrightarrow \text{Irr}_{d,n+1}(\mathbb{C})$ given by forgetting the $(n+1)$-th variable. This inclusion is an embedding of a closed subspace, and hence a proper map.

Theorem 1.2. For $n, d > 1$ and for any $i < \frac{2n}{d-1} - 1$, the natural inclusion $\text{Irr}_{d,n}(\mathbb{C}) \hookrightarrow \text{Irr}_{d,n+1}(\mathbb{C})$ induces an isomorphism

$$H^i_!(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}) \cong H^i_!(\text{Irr}_{d,n+1}(\mathbb{C}); \mathbb{Q}).$$

$Ir_{d,n}(\mathbb{C})$ is a complex manifold and satisfies Poincaré duality for compactly supported cohomology. Theorem 1.2 equivalently says that the cohomology of $\text{Irr}_{d,n}(\mathbb{C})$ stabilizes in fixed codimensions as $n \to \infty$. Hence, Theorem 1.1 and Theorem 1.2 cover different ranges of the cohomology of $\text{Irr}_{d,n}(\mathbb{C})$.

Unlike Theorem 1.1, Theorem 1.2 only shows cohomological stability without telling us what the stable cohomology is. In the last part of the paper, we will study the limit

$$b_i(d) := \lim_{n \to \infty} \dim H^i_!(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}).$$

We prove that $b_i(d) = 0$ when $i \leq 2d$ and $d \geq 2$ (Corollary 6.2). However, $b_i(d)$ are generally nonzero when $i$ is large enough. As examples, in the Appendix we compute $b_i(d)$ for all $i$ in the range $d \leq 3$ and showed that $b_{11}(4) = 1$.

Our methods are topological and do not use the Grothendieck-Lefschetz trace formula. We will consider a stratification of the space of polynomials according to how they factor (Section 3), and then analyze the spectral sequence induced by the stratification (Section 4, 5 and 6).

Remark 1.2 (Related works). Hyde (Theorem 1.22 in [7]) recently proved that the compactly supported Euler characteristic of $\text{Irr}_{d,n}(\mathbb{C})$ is 0 when $d > 1$. Note that $\chi_c(\text{Irr}_{d,n}(\mathbb{C})) = \chi(\text{Irr}_{d,n}(\mathbb{C}))$ by Poincaré duality. Since the stable cohomology of $\text{Irr}_{d,n}(\mathbb{C})$ as in Theorem 1.1 is supported in even degrees, we expect $\text{Irr}_{d,n}(\mathbb{C})$ to have nonzero odd cohomology groups in the unstable range.

Tommasi [8] proved that the rational cohomology of the space $X_{d,n}$ of nonsingular complex homogeneous polynomials of degree $d$ in $n + 1$ variables stabilizes as $d \to \infty$. Since the defining equation of any nonsingular hypersurface is irreducible, $\text{Irr}_{d,n+1}(\mathbb{C})$ contains the projectivized $X_{d,n}/\mathbb{C}^\times$. Comparing Theorem 1.1 and Tommasi’s result, we see that even though $\text{Irr}_{d,n}(\mathbb{C})$ and $X_{d,n}/\mathbb{C}^\times$ both satisfy cohomological stability as $d \to \infty$, their stable cohomology groups are different. Tommasi’s theorem implies that the stable cohomology of $X_{d,n}/\mathbb{C}^\times$ is isomorphic to the cohomology of $\text{PGL}_{n+1}(\mathbb{C})$ which is generated by classes with odd degrees; in contrast, Theorem 1.1 tells us that the stable cohomology of $\text{Irr}_{d,n}(\mathbb{C})$ is supported in even degrees.

The theme of this paper is close to that of Farb-Wolofson-Wood [4], where they proved surprising coincidences in the Poincaré series of certain apparently unrelated spaces, which were predicted by the corresponding point-counting results over finite fields (Theorem 1.2 in [4]). Our Theorem 1.1 and 1.2 as well as the reasoning that leads us to discover them, provide another example where the Grothendieck-Lefschetz trace formula, despite not playing any role in the proofs, can still provide heuristics leading to plausible conjectures, which are then settled by topological methods.

Acknowledgement

The author would like to thank Ronno Das, Nir Gadish, and Trevor Hyde for helpful conversations.
2. From counting to cohomology

We will briefly explain the heuristic that leads us to ask Question 1 and 2 from Carlitz’ and Hyde’s counting results. Our reasoning here was inspired by the work of Church-Ellenberg-Farb [2].

For $X$ a variety over $\mathbb{Z}$, the Grothendieck-Lefschetz trace formula gives

$$|X(\mathbb{F}_q)| = \sum_i (-1)^i \text{Trace} \left( \text{Frob}_q : H^i_{\text{ét},c}(X/\mathbb{F}_q; \mathbb{Q}_\ell) \right)$$

(2.1)

where $X(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$-points on $X$, and the right hand side involves the trace of Frobenius acting on the compactly supported étale cohomology of $X$ over $\overline{\mathbb{F}}_q$ with $\mathbb{Q}_\ell$-coefficient for $\ell$ a prime not dividing $q$. Deligne proved that all the eigenvalues of Frobenius on $H^i_{\text{ét},c}(X; \mathbb{Q}_\ell)$ have absolute values no more than $q^{i/2}$ (Théorème 2 in [3]).

For the sake of heuristic reasoning, let us suppose that there is a variety $X_{d,n}$ over $\mathbb{Z}$ such that $X_{d,n}(\mathbb{F}_q) = \text{Irr}_{d,n}(\mathbb{F}_q)$ and $X_{d,n}(\mathbb{C}) = \text{Irr}_{d,n}(\mathbb{C})$. Since Hyde proved that $|\text{Irr}_{d,n}(\mathbb{F}_q)|$ is a polynomial in $q$, the Grothendieck-Lefschetz trace formula (2.1) together with Deligne’s bounds would tell us that roughly the low $q$-powers in the polynomial $|\text{Irr}_{d,n}(\mathbb{F}_q)|$ come from $H^i_{\text{ét}}(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q})$ for $i$ small. Since Hyde [1,2] proved that the low $q$-powers in $|\text{Irr}_{d,n}(\mathbb{F}_q)|$ converge as $n \to \infty$, one would expect that $H^i_{\text{ét}}(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q})$ should stabilize as $n$ increases. Similarly, Carlitz [1,1] proved that the high $q$-powers in $|\text{Irr}_{d,n}(\mathbb{F}_q)|$ converge as $d \to \infty$. One would therefore expect that $H^i(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q})$ should stabilize as $d$ increases, after applying Poincaré duality. These are the reasons why we ask Question 1 and 2 and expect positive answers.

Remark 2.1 (Counting geometrically irreducible polynomials). It turns out that the variety $X_{d,n}$ satisfying our assumptions above does not exist. However, there does exist a variety $Y_{d,n}$ over $\mathbb{Z}$ such that $Y_{d,n}(\mathbb{C}) = \text{Irr}_{d,n}(\mathbb{C})$ and $Y_{d,n}(\mathbb{F}_q)$ is the set of geometrically irreducible polynomials, namely, polynomials over $\mathbb{F}_q$ that cannot be written as a nontrivial product of polynomials over $\mathbb{F}_q$. Moreover, $|Y_{d,n}(\mathbb{F}_q)|$ can be expressed in terms of $|\text{Irr}_{d/e,n}(\mathbb{F}_{q^e})|$ for $e$ divisors of $d$. Hyde (personal communication) verified that $|Y_{d,n}(\mathbb{F}_q)|$ satisfies the same convergence phenomena as $|\text{Irr}_{d,n}(\mathbb{F}_q)|$. Therefore, one can make the heuristic reasoning above a rigorous argument if one replaces $|\text{Irr}_{d,n}(\mathbb{F}_q)|$ by $|Y_{d,n}(\mathbb{F}_q)|$, although we will not adopt this approach in the present paper.

3. Preliminary lemmas

We first prove some preliminary results that will be used later in the paper. The results we collect here can be viewed as topological analogs of Lemma 2.1 in [6].

Consider the space

$$\text{Poly}_{\leq d,n}(\mathbb{C}) := \{\text{nonzero complex polynomials in } n \text{ variables with total degree } \leq d\}/\mathbb{C}^\times.$$

Note that $\text{Poly}_{\leq d,n}(\mathbb{C}) = \mathbb{C}P^{(d+n)} - 1$ because there are $\binom{d+n}{n}$ many monomials of degree $\leq d$ in $n$ variables. Next define $\text{Poly}_{d,n}(\mathbb{C}) := \text{Poly}_{\leq d,n}(\mathbb{C}) \setminus \text{Poly}_{\leq d-1,n}(\mathbb{C})$. This is the space of normalized multivariate polynomials with total degree $d$.

Lemma 3.1. For any $d$ and $n$, we have a homeomorphism:

$$\text{Poly}_{d,n}(\mathbb{C}) \cong \mathbb{C}(d+n\times) \times \mathbb{C}P^{(d+n-1)} - 1.$$

Thus, $\text{Poly}_{d,n}(\mathbb{C})$ is homotopy equivalent to $\mathbb{C}P^{(d+n-1)} - 1$.

Proof. Observe that any $f \in \text{Poly}_{d,n}(\mathbb{C})$ can be written uniquely as

$$f = f_d + f_{<d}$$

where $f_d$ is a homogeneous polynomial of degree $d$ (up to scalar) and $f_{<d}$ is an arbitrary polynomial of degree $< d$. The map $f \mapsto (f_{<d}, f_d)$ gives the isomorphism. \(\square\)
Define
\[ \text{Red}_{d,n}(\mathbb{C}) := \{ f \in \text{Poly}_{d,n}(\mathbb{C}) : f \text{ is reducible} \} \]
which is a closed subspace of \( \text{Poly}_{d,n}(\mathbb{C}) \) with open complement \( \text{Irr}_{d,n}(\mathbb{C}) \). We have a long exact sequence:
\[ \cdots \to H^*_c(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Z}) \to H^*_c(\text{Poly}_{d,n}(\mathbb{C}); \mathbb{Z}) \to H^*_c(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Z}) \to \cdots \] (3.1)

Every \( f \in \text{Poly}_{d,n}(\mathbb{C}) \) can be factorized uniquely into a product of irreducible polynomials up to scalars \( f = f_1 f_2 \cdots f_l \), which gives a unique partition \( \lambda_f \) of the integer \( d = \deg f \) by
\[ \lambda_f : \deg(f_1) + \deg(f_2) + \cdots + \deg(f_l) = d. \]
For any partition \( \lambda \) of \( d \) (written as \( \lambda \vdash d \) in the future), we define the following subspace of \( \text{Poly}_{d,n}(\mathbb{C}) \):
\[ T_{\lambda,n} := \{ f \in \text{Poly}_{d,n}(\mathbb{C}) : \lambda_f = \lambda \}. \]

We use \( \text{Sym}^m X \) to denote the \( m \)-th symmetric power of a topological space \( X \). So \( \text{Sym}^m X := X^m/S_m \) where the symmetric group \( S_m \) acts on \( X^m \) by permuting the coordinates. For \( d \vdash \lambda \) and \( j \in \mathbb{Z}_{>0} \), we will let \( m_j(\lambda) \) denote the multiplicity of \( j \) in \( \lambda \). Every polynomial \( f \in T_{\lambda,n} \) can be factorized uniquely into \( \prod_{j=1}^d f_{j,1} f_{j,2} \cdots f_{j,m_j} \) where each \( f_{j,k} \in \text{Irr}_{j,n}(\mathbb{C}) \), up to reordering. Thus, we have
\[ T_{\lambda,n} \cong \prod_{j=1}^d \text{Sym}^{m_j(\lambda)}(\text{Irr}_{j,n}(\mathbb{C})). \] (3.2)
The unique factorization of polynomials gives the following decomposition of \( \text{Poly}_{d,n}(\mathbb{C}) \) into disjoint subsets:
\[ \text{Poly}_{d,n}(\mathbb{C}) = \bigcup_{\lambda \vdash d, |\lambda| \geq 2} T_{\lambda,n} \] (3.3)
Let \( (d) \) denote the trivial partition with a single part. Notice that \( T_{(d),n} = \text{Irr}_{d,n}(\mathbb{C}) \).

We will focus on the decomposition of the space of reducible polynomials:
\[ \text{Red}_{d,n}(\mathbb{C}) = \bigcup_{\lambda \vdash d, |\lambda| \geq 2} T_{\lambda,n} \] (3.4)
where \( |\lambda| := \sum_j m_j(\lambda) \) denote the total number of parts in the partition \( \lambda \).

**Lemma 3.2.** There is a spectral sequence
\[ E^1_{p,q} = \bigoplus_{\lambda : \lambda \vdash d, |\lambda| = d - p + \lambda} H^p_c(T_{\lambda,n}; \mathbb{Z}) \Rightarrow H^p_c(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Z}). \] (3.5)

**Proof.** Consider the following increasing filtration of \( \text{Red}_{d,n}(\mathbb{C}) \):
\[ \emptyset = F_0 \subset F_1 \subset \cdots \subset F_{d-1} = \text{Red}_{d,n}(\mathbb{C}) \quad \text{where each} \quad F_p := \bigcup_{\lambda : |\lambda| \geq d + 1 - p} T_{\lambda,n}. \] (3.6)
We claim that each \( F_p \) is a closed subspace of \( \text{Poly}_{d,n}(\mathbb{C}) \). In fact, we have \( T_{\lambda,n} \subseteq F_{\mu,n} \) if \( \lambda \) is finer than or equal to \( \mu \). To see this, notice that if a sequence of polynomials \( f_n \in T_{\mu,n} \) converges to a limit \( f \), then \( f \) can be factorized in the same pattern as each \( f_n \) because being a product is a closed condition. However, the irreducible factors of \( f_n \) might become reducible in the limit because being irreducible is an open condition. Hence, \( \lambda_f \) is finer than or equal to \( \mu \).

We will abbreviate the compactly supported cochain complex \( C_*^c(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Z}) \) simply as \( C^* \). The increasing filtration \( \{ F_p \} \) of \( \text{Red}_{d,n}(\mathbb{C}) \) induces a decreasing filtration \( \{ G^p C^* \} \) of \( C^* \):
\[ C^* = G^0 C^* \supset G^1 C^* \supset \cdots \supset G^{d-1} C^* = 0 \quad \text{where each} \quad G^p C^* := C^c(\text{Red}_{d,n}(\mathbb{C}) \setminus F_p; \mathbb{Z}). \]
Since each \( F_p \) is a closed subspace of \( \text{Red}_{d,n}(\mathbb{C}) \), we have
\[ \frac{G^p C^*}{G^{p+1} C^*} \cong C^c(F_{p+1} \setminus F_p; \mathbb{Z}). \]
Thus, the filtered complex $\{G_p^\lambda C^*\}$ induces a spectral sequence with $E_1$-page:

$$E_1^{p,q} = H_c^{p+q}(F_{p+1} \setminus F_p; \mathbb{Z}) \Rightarrow H_c^{p+d}(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Z}).$$

Finally, by (3.6) we have

$$\mathcal{F}_{p+1} \setminus \mathcal{F}_p = \bigcup_{\lambda: |\lambda| = d-p} T_{\lambda,n}. \quad (3.7)$$

For any two distinct partitions $\lambda$ and $\mu$ of equal size $d-p$, we have $T_{\mu,n} \cap T_{\lambda,n} = \emptyset$ because it is impossible that $\lambda$ is finer than $\mu$. Thus, the set-theoretical disjoint union (3.7) is actually a disjoint union of topological spaces. Hence, we obtain the spectral sequence (3.5).

\[\Box\]

4. Proof of Theorem 1.1

Theorem 1.1 in the Introduction will follow from Theorem 4.1 below together with Lemma 3.1.

**Theorem 4.1.** For $n > 1$, the inclusion $\text{Irr}_{d,n}(\mathbb{C}) \hookrightarrow \text{Poly}_{d,n}(\mathbb{C})$ induces an isomorphism

$$H^i(\text{Irr}_{d,n}(\mathbb{C}), \mathbb{Z}) \cong H^i(\text{Poly}_{d,n}(\mathbb{C}), \mathbb{Z})$$

when $i \leq 2\left(\binom{d+n-1}{n-1} - n - 1\right)$.

**Remark 4.1.** In [1], Carlitz obtained his result by showing that $|\text{Irr}_{d,n}(\mathbb{F}_q)| \sim |\text{Poly}_{d,n}(\mathbb{F}_q)|$ as $d \to \infty$ when $n > 1$. Theorem 4.1 is a topological analog of Carlitz’ observation that “when the number of indeterminates is greater than one we find that almost all polynomials are irreducible” ([1], Section 1). The assumption $n > 1$ is needed in our proof below.

**Proof of Theorem 4.1.** Since $\text{Irr}_{d,n}(\mathbb{C}) \hookrightarrow \text{Poly}_{d,n}(\mathbb{C})$ is an inclusion of complex (hence orientable) manifolds of equal complex dimension $\left(\binom{d+n}{n} - 1\right)$, by Poincaré duality, in order to prove Theorem 4.1, it suffices to prove that the inclusion $\text{Irr}_{d,n}(\mathbb{C}) \hookrightarrow \text{Poly}_{d,n}(\mathbb{C})$ induces an isomorphism on compactly supported cohomology:

$$H^i_c(\text{Irr}_{d,n}(\mathbb{C}), \mathbb{Z}) \cong H^i_c(\text{Poly}_{d,n}(\mathbb{C}), \mathbb{Z})$$

when $i \geq 2\left(\binom{d+n}{n} - 1\right) - 2\left(\binom{d+n-1}{n-1} - n - 1\right) = 2\left(\binom{d+n-1}{n} + n\right)$. By the long exact sequence (3.1), it suffices to prove the following proposition:

**Proposition 4.2.** For $n > 1$, we have $H^i_c(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Z}) = 0$ when $i \geq 2\left(\binom{d+n-1}{n} + n\right) - 1$.

Before proving Proposition 4.2, we will first prove the following lemma:

**Lemma 4.3.** For any partition $\lambda$ of $d$ such that $|\lambda| \geq 2$, we have $\dim_{\mathbb{C}}(T_{\lambda,n}) \leq \binom{d+n-1}{n} + n - 1$.

**Proof.** $\text{Irr}_{d,n}(\mathbb{C})$ is an open subset of $\text{Poly}_{d,n}(\mathbb{C})$ and thus is a manifold of complex dimension $\left(\binom{d+n}{n} - 1\right)$. Hence, $\text{Sym}^m(\text{Irr}_{d,n}(\mathbb{C}))$ is a orbifold (i.e. a manifold quotient by a finite group action) of complex dimension $m\left(\binom{d+n}{n} - 1\right)$. By (3.2), each $T_{\lambda,n}$ is also an orbifold with dimension

$$\dim_{\mathbb{C}}(T_{\lambda,n}) = \sum_{j=1}^{d} m_j(\lambda) \left(\binom{j+n}{n} - 1\right). \quad (4.1)$$

Since the function $\binom{j+n}{n} - 1$ is strictly convex in $j$ when $n > 1$, we have $\dim_{\mathbb{C}}(T_{\lambda,n}) < \dim_{\mathbb{C}}(T_{\mu,n})$ if $\lambda$ is strictly finer than $\mu$. Therefore, $\dim_{\mathbb{C}}(T_{\lambda,n})$ is maximized at some partition $\lambda$ of size exactly
2. Hence, it suffices to consider $\lambda$ to be of the form $k + (d - k)$ for some integer $k = 1, \ldots, d - 1$. For such $\lambda$, we have

$$\dim_{\mathbb{C}}(T_{\lambda, n}) = \left(\frac{k + n}{n}\right) + \left(\frac{d - k + n}{n}\right) - 2 =: f(k).$$

By checking its second derivative, the function $f(k)$ is strictly convex for $k \in [1, d - 1]$ and thus the only possible local maximum occur at the two endpoints. Hence, for any $k \in [1, d - 1]$, we have $f(k) \leq f(1) = f(d - 1)$.

**Proof of Proposition 4.3.** Consider the spectral sequence in Lemma 3.2:

$$E_1^{p,q} = \bigoplus_{\lambda: \lambda-d,|\lambda|=d-p \geq 2} H_c^{p+q}(T_{\lambda, n}; \mathbb{Z}) \implies H_c^{p+q}(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Z}).$$

Lemma 4.3 implies that $E_1^{p,q} = 0$ when $p + q > 2\left(\frac{d + n - 1}{2} + n - 1\right)$. Thus, we have

$$H_c^{i}(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Z}) = 0$$

when $i > 2\left(\frac{(d + n - 1)}{2} + n - 1\right)$. \(\square\)

Theorem 4.1 now follows from Proposition 4.2. \(\square\)

5. **Proof of Theorem 1.2**

Again as in the proof of Theorem 4.1, the decomposition $\text{Irr}_{d,n}(\mathbb{C}) = \text{Poly}_{d,n}(\mathbb{C}) \setminus \text{Red}_{d,n}(\mathbb{C})$ gives the following long exact sequence

$$\cdots \to H_c^i(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}) \to H_c^i(\text{Poly}_{d,n}(\mathbb{C}); \mathbb{Q}) \to H_c^i(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Q}) \to \cdots$$

By Lemma 3.1 we have $H_c^i(\text{Poly}_{d,n}(\mathbb{C}); \mathbb{Q}) = 0$ when $i < 2\left(\frac{n + d - 1}{2}\right)$. Thus, the long exact sequence gives the following lemma:

**Lemma 5.1.** For any $n$ and $d$, when $i < 2\left(\frac{n + d - 1}{2}\right) - 1$, the natural connecting homomorphism is an isomorphism:

$$H_c^i(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Q}) \cong H_c^{i+1}(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}). \tag{5.1}$$

When $d > 1$, we have

$$\frac{2n}{d - 1} - 2 < \frac{2(n + d - 1)}{d - 1} - 1 \leq 2\left(\frac{n + d - 1}{d - 1}\right) - 1 = 2 \cdot \frac{n + d - 1}{d - 1} \cdot \frac{n + d - 2}{d - 2} \cdots \frac{n}{1} - 1$$

Thus, for any $i$ in the range as stated in Theorem 1.2, the isomorphism (5.1) holds.

Since $\text{Irr}_{d,n}(\mathbb{C})$ is connected (by Theorem 1.1) and noncompact, it has vanishing $H_c^0$. Theorem 1.2 is already true for $i = 0$. To prove Theorem 1.2 for $i > 0$, it suffices to prove the following proposition by Lemma 5.1.

**Proposition 5.2.** For $n, d > 1$ and for any $i < \frac{2n}{d - 1} - 2$, the inclusion $\text{Red}_{d,n}(\mathbb{C}) \hookrightarrow \text{Red}_{d,n+1}(\mathbb{C})$ induces an isomorphism

$$H_c^i(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Q}) \cong H_c^{i}(\text{Red}_{d,n+1}(\mathbb{C}); \mathbb{Q}).$$

**Proof.** Consider the spectral sequence in Lemma 3.2 tensored with $\mathbb{Q}$:

$$E_1^{p,q} = \bigoplus_{\lambda: \lambda-d,|\lambda|=d-p \geq 2} H_c^{p+q}(T_{\lambda, n}; \mathbb{Q}) \implies H_c^{p+q}(\text{Red}_{d,n}(\mathbb{C}); \mathbb{Q}). \tag{5.2}$$

Since the inclusion $\text{Red}_{d,n}(\mathbb{C}) \hookrightarrow \text{Red}_{d,n+1}(\mathbb{C})$ preserves the filtration (3.6), it induces a map of the spectral sequences. Proposition 5.2 will follow if the inclusion induces isomorphisms on $E_1^{p,q}$ when $p + q < \frac{2n}{d - 1} - 2 + 1$. Thus, it suffices to prove the following proposition:
**Proposition 5.3.** Suppose $n, d > 1$ and $|\lambda| \geq 2$. For any $i < \frac{2n}{d-1} - 1$, the inclusion $T_{\lambda,n} \hookrightarrow T_{\lambda,n+1}$ induces an isomorphism

$$H^i_c(T_{\lambda,n}; \mathbb{Q}) \cong H^i_c(T_{\lambda,n+1}; \mathbb{Q}).$$

**Proof.** The proof of Proposition 5.3 will proceed in three steps.

**Step 1.** We first prove the following general results about graded vector spaces.

**Lemma 5.4.** Suppose $f : A \to B$ and $g : C \to D$ are maps of graded vector spaces. If for any $i \leq r$, the maps $f : A_i \congrightarrow B_i$ and $g : C_i \congrightarrow D_i$ are isomorphisms on the $i$-th graded pieces, then the following maps

$$(A \otimes C)_i \xrightarrow{f \otimes g} (B \otimes D)_i$$

$$((A \otimes^m)^s)_i \longrightarrow (B \otimes^m)^s_i$$

are also isomorphisms for any $i \leq r$.

**Proof.** For any $i \leq r$, we have

$$(A \otimes C)_i = \bigoplus_{s+t=i} A_s \otimes C_t \xrightarrow{f \otimes g} \bigoplus_{s+t=i} B_s \otimes D_t = (B \otimes D)_i$$

since each $s$ and $t$ in the summand are no more than $i$ and hence $r$. Moreover, applying the reasoning above inductively on $m$, we have

$$(A \otimes^m)_i \xrightarrow{f \otimes^m} (B \otimes^m)_i \quad \text{for} \quad i \leq r.$$

Observe that the isomorphism is equivariant with respect to the action of $S_m$. Taking the $S_m$-invariants, we obtain the second claim.

**Step 2.** Next we apply Lemma 5.4 to study the compactly supported cohomology of symmetric powers.

**Lemma 5.5.** Suppose $X$ is a closed subspace of $Y$ such that the inclusion $\text{inc} : X \hookrightarrow Y$ induces an isomorphism

$$\text{inc}^* : H^i_c(Y; \mathbb{Q}) \congrightarrow H^i_c(X; \mathbb{Q})$$

for any $i \leq r$. Then for any natural number $m$, the inclusion $\text{inc} : \text{Sym}^m(X) \hookrightarrow \text{Sym}^m(Y)$ also induces an isomorphism

$$\text{inc}^* : H^i_c(\text{Sym}^m(Y); \mathbb{Q}) \congrightarrow H^i_c(\text{Sym}^m(X); \mathbb{Q})$$

for any $i \leq r$.

**Proof.** Since $X$ is a closed subspace of $Y$, the symmetric power $\text{Sym}^m(X)$ is also a closed subspace of $\text{Sym}^m(Y)$. Hence the inclusion map $\text{Sym}^m(X) \hookrightarrow \text{Sym}^m(Y)$ is proper and induces maps on cohomology groups with compact support.

Moreover, we have

$$H^*(\text{Sym}^mY; \mathbb{Q}) = H^*(Y^m/S_m; \mathbb{Q}) \cong H^*(Y^m; \mathbb{Q})^S_m \cong (H^*_c(Y; \mathbb{Q}) \otimes^m)^S_m$$

where the second isomorphism is the transfer homomorphism. Lemma 5.5 now follows by applying Lemma 5.4 to $A = H^*_c(\text{Sym}^mY; \mathbb{Q})$ and $B = H^*_c(\text{Sym}^mX; \mathbb{Q})$.

**Step 3.** Finally, we will prove Proposition 5.3 by induction on $d > 1$. The base case is when $d = 2$. The only nontrivial partition $\lambda + 2$ is $1+1 = 2$. In this case, $T_{\lambda,n} = \text{Sym}^2(\text{Irr}_{1,n}) = \text{Sym}^2(CP^{n-1})$. Similarly, $T_{\lambda,n+1} = \text{Sym}^2(CP^n)$. Since the natural inclusion $CP^{n-1} \hookrightarrow CP^n$ induces isomorphisms on $H^*_c$ for $i \leq 2(n-1)$, by Lemma 5.5 the inclusion $T_{\lambda,n} \hookrightarrow T_{\lambda,n+1}$ also induces isomorphisms on $H^*_c$ for $i \leq 2(n-1)$. 
For induction, suppose that for any nontrivial partition \( \lambda \vdash d' \) where \( d' < d \), and for any \( i < \frac{2n}{d-1} - 1 \), we have
\[
H^*_c(T_{\lambda,n}; \mathbb{Q}) \cong H^*_c(T_{\lambda,n+1}; \mathbb{Q}).
\]
By the argument above, this implies that for any \( i < \frac{2n}{d-1} - 1 \), we have isomorphisms
\[
H^*_c(\text{Irr}_{d',n}(\mathbb{C}); \mathbb{Q}) \cong H^*_c(\text{Irr}_{d',n+1}(\mathbb{C}); \mathbb{Q}). \tag{5.3}
\]
We want to prove the statement for any nontrivial partition \( \lambda \) of \( d \). Since \( \lambda \vdash d \) is not a singleton, each part of \( \lambda \) must have length at most \( d - 1 \). Compare the following two isomorphisms of graded vector spaces:
\[
H^*_c(T_{\lambda,n}; \mathbb{Q}) \cong \bigotimes_{j=1}^{d-1} H^*_c(\text{Sym}^{m_j}(\lambda) \text{Irr}_{j,n}(\mathbb{C}); \mathbb{Q}) \tag{5.4}
\]
\[
H^*_c(T_{\lambda,n+1}; \mathbb{Q}) \cong \bigotimes_{j=1}^{d-1} H^*_c(\text{Sym}^{m_j}(\lambda) \text{Irr}_{j,n+1}(\mathbb{C}); \mathbb{Q}) \tag{5.5}
\]
Let \( r := \frac{2n}{d-1} - 1 \). For each \( j \leq d-1 \), if \( i < r \) then \( i < \frac{2n}{j-1} - 1 \), and thus by the induction hypothesis (5.3) applied to \( d' = j < d \), the inclusion \( \text{Irr}_{j,n}(\mathbb{C}) \hookrightarrow \text{Irr}_{j,n+1}(\mathbb{C}) \) induces an isomorphism
\[
H^*_c(\text{Irr}_{j,n}(\mathbb{C}); \mathbb{Q}) \cong H^*_c(\text{Irr}_{j,n+1}(\mathbb{C}); \mathbb{Q}).
\]
Thus, if we compare (5.4) and (5.5) using Lemma 5.4 and Lemma 5.5 we have that for any \( i < r \)
\[
H^*_c(T_{\lambda,n}; \mathbb{Q}) \cong H^*_c(T_{\lambda,n+1}; \mathbb{Q}).
\]
Proposition 5.3 follows by induction. \( \square \)

6. A VANISHING THEOREM

**Theorem 6.1.** For \( d, n > 1 \), when \( k \leq 2d \), we have
\[
H^*_c(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}) = 0.
\]

Taking the limit \( n \to \infty \), we obtain the following corollary.

**Corollary 6.2.** For \( d > 1 \), when \( k \leq 2d \), we have
\[
b_k(d) := \lim_{n \to \infty} \dim H^*_c(\text{Irr}_{d,n}(\mathbb{C}); \mathbb{Q}) = 0.
\]

**Proof of Theorem 6.1.** We will prove Theorem 6.1 in three steps.

**Step 1.** We first collect some general results about graded vector spaces.

**Lemma 6.3.** Suppose \( A \) and \( B \) are graded vector spaces. If \( A_i = 0 \) for any \( i < a \), and \( B_j = 0 \) for any \( j < b \), then
\[
(i) \quad (A \otimes B)_k = 0 \text{ for any } k < a + b, \quad (ii) \quad (A^\otimes m)_k = 0 \text{ for any } k < ma.
\]

**Proof.** Suppose (i) is false: there exists some \( k < a + b \) such that \( (A \otimes B)_k = \bigoplus_{i+j=k} A_i \otimes B_j \neq 0 \).

There exist some \( i, j \) such that \( i + j = k \) and \( A_i \neq 0 \) and \( B_j \neq 0 \), which implies that \( i \geq a \) and \( j \geq b \), and thus \( i + j \geq a + b \), contradicting the assumption that \( i + j = k < a + b \).

Applying (i) inductively on \( m \), we obtain that \( (A^\otimes m)_k = 0 \) for any \( k < ma \). Thus the \( S_m \)-invariant subspace must also be zero. \( \square \)

**Step 2.** We will inductively define a function \( r : \mathbb{N} \to \mathbb{N} \) and compute its value.
Definition 6.1. For each positive integer $d$, define $r(d)$ inductively by

$$r(1) = 2$$

$$\forall d > 1, \quad r(d) = 1 + \min \left\{ r(\lambda) : \lambda \vdash d, |\lambda| \geq 2 \right\},$$

where for each $\lambda \vdash d$ such that $|\lambda| \geq 2$, we define $r(\lambda) := \sum_{j=1}^{d-1} m_j(\lambda)r(j)$.

Proposition 6.4. Suppose that $d > 1$.

(a) For any $\lambda \vdash d$ such that $|\lambda| \geq 2$, we have $r(\lambda) = 2d + |\lambda| - m_1(\lambda)$.

(b) The minimum $r(\lambda)$ is uniquely achieved at $\lambda = \hat{1}$ where $\hat{1}$ stands for the partition $d = 1 + \cdots + 1$.

(c) $r(d) = 2d + 1$.

Proof. We will prove the three statements by induction on $d \geq 2$. For the base case when $d = 2$, the three statements are easily verified since $1 + 1$ is the only nontrivial partition of 2.

For induction, we consider the case when $d > 2$, assuming the three statements all hold for any $d' < d$. For any nontrivial partition $\lambda \vdash d$ with $|\lambda| \geq 2$, we have

$$r(\lambda) := \sum_{j=1}^{d-1} m_j(\lambda)r(j)$$

$$= m_1(\lambda) \cdot 2 + \sum_{j=2}^{d-1} m_j(\lambda)(2j + 1) \quad \text{by induction hypothesis (3)}$$

$$= \sum_{j=1}^{d-1} m_j(\lambda) \cdot 2j + \sum_{j=2}^{d-1} m_j(\lambda)$$

$$= 2d + |\lambda| - m_1(\lambda)$$

Thus, (a) is verified. (b) and (c) follow immediately from (a). □

Step 3. We will prove the following vanishing result in a range defined by the function $r$.

Proposition 6.5. For any $d \geq 1$ and $n \geq 2$,

(1) for any partition $\lambda \vdash d$ such that $|\lambda| \geq 2$, we have

$$H^k_c(T_{\lambda,n}; \mathbb{Q}) = 0 \quad \text{when} \quad k < r(\lambda)$$

(2)

$$H^k_c(Irr_{d,n}(\mathbb{C}); \mathbb{Q}) = 0 \quad \text{when} \quad k < r(d).$$

Proof. We will prove both statements by induction on $d$. For the base case when $d = 1$, part (1) is vacuously true. We have

$$\text{Irr}_{1,n}(\mathbb{C}) = \text{Poly}_{1,n}(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C}P^{n-1}$$

where the second homeomorphism comes from Lemma 3.1. Part (2) is also verified.

We consider the case when $d \geq 2$ for the induction, assuming both (1) and (2) hold for any $d' < d$. Since $\lambda$ has at least two parts, each part has size at most $d - 1$. Recall that 3.2 gives:

$$T_{\lambda,n} \cong \prod_{j=1}^{d-1} \text{Sym}^{m_j(\lambda)}(\text{Irr}_{j,n}(\mathbb{C})).$$

By induction hypothesis part (2), for each $j \leq d - 1$, we have that $H^k_c(\text{Irr}_{j,n}(\mathbb{C}); \mathbb{Q}) = 0$ when $k < r(j)$. Now we briefly digress to prove the following general results about symmetric powers.
Lemma 6.6. If $H^k(Y; \mathbb{Q}) = 0$ for any $k < r$, then $H^k(\text{Sym}^m Y; \mathbb{Q}) = 0$ for any $k < mr$.

Proof. Observe that
\[ H^*_c(\text{Sym}^m Y; \mathbb{Q}) = H^*_c(Y \times m/S_m; \mathbb{Q}) \cong H^*_c(Y \times m; \mathbb{Q})^{S_m} \cong (H^*_c(Y; \mathbb{Q}) \otimes m)^{S_m}. \]

Apply part (ii) of Lemma 6.3. □

Thus, by Lemma 6.6, we have that $H^k_c(\text{Sym}^m Y; \mathbb{Q}) = 0$ when $k < m r_j$ for any $j$. By Lemma 6.3 part (i), we have $H^k_c(T_{\lambda, n}; \mathbb{Q}) = 0$ (6.2) when $k < \sum_{j=1}^{d-1} m_j(\lambda)r(j)$. Part (1) is verified.

To prove part (2), we consider the spectral sequence (3.5)
\[ E_1^{p, q} = \bigoplus_{\lambda: \lambda \vdash d, |\lambda| = d - p - 2} H^{p+q}_c(T_{\lambda, n}; \mathbb{Q}) \Rightarrow H^{p+q}_c(\text{Red}_{d, n}(\mathbb{C}); \mathbb{Q}) \]

By (6.2), we know that $E_1^{p, q} = 0$ in the range when $p + q < r(d) - 1$. Thus, when $k < r(d) - 1$, $H^k_c(\text{Red}_{d, n}(\mathbb{C}); \mathbb{Q}) = 0$.

Since $n, d \geq 2$, by Proposition 6.4 we have $r(d) - 1 = 2d < 2(n + d - 1) - 1$. Thus, by Lemma 5.1 when $i < r(d) - 1$, we have $H^{i+1}_c(\text{Irr}_{d, n}(\mathbb{C}); \mathbb{Q}) \cong H^i_c(\text{Red}_{d, n}(\mathbb{C}); \mathbb{Q}) = 0$.

Part (2) is verified. □

Finally, combining part (2) of Proposition 6.5 and part (c) of Proposition 6.4, we obtain Theorem 6.1. □

7. Appendix: Computation for $d \leq 4$

In this appendix, we consider the stable cohomology in Theorem 1.2 more precisely, the limit
\[ b_i(d) := \lim_{n \to \infty} \dim H^i_c(\text{Irr}_{d, n}(\mathbb{C}); \mathbb{Q}) \] (7.1)

for $d \leq 4$. Theorem 1.2 tells us that the limit exists. The purpose of our computations here is to illustrate that the stable cohomology in Theorem 1.2 are generally nonzero despite the vanishing result in Theorem 6.1 and that the spectral sequence (3.5) which is central in the previous proofs has nontrivial differentials, even in the stable range. To keep this appendix brief, we will only sketch the computations, highlighting the analysis of differentials in the spectral sequence.

As in Theorem 1.2 there is a closed embedding (hence a proper map) $\text{Poly}_{d, n}(\mathbb{C}) \to \text{Poly}_{d, n+1}(\mathbb{C})$ for each $n$. We define the following direct limits
\[ \text{Poly}_d(\mathbb{C}) := \lim_{\to} \text{Poly}_{d, n}(\mathbb{C}) \]
\[ \text{Irr}_d(\mathbb{C}) := \lim_{\to} \text{Irr}_{d, n}(\mathbb{C}) \]
\[ \text{Red}_d(\mathbb{C}) := \lim_{\to} \text{Red}_{d, n}(\mathbb{C}) \]
\[ T_{\lambda} := \lim_{\to} T_{\lambda, n} \quad \text{for each } \lambda \vdash d. \]

Since compactly supported cohomology preserves limits, the stable cohomology can be expressed as:
\[ b_i(d) = \dim H^i_c(\text{Red}_d(\mathbb{C}); \mathbb{Q}). \]
All cohomology considered in this section will be over \( \mathbb{Q} \). We will therefore suppress the \( \mathbb{Q} \)-coefficients from our notation. We will encode our computation of \( H_c^*(\text{Irr}_d(\mathbb{C})) \) into a Poincaré series:

\[
P_d(t) := \sum_i b_i(d)t^i.
\]

**d=1.** We have \( \text{Irr}_{1,n}(\mathbb{C}) = \text{Poly}_{1,n}(\mathbb{C}) = \mathbb{C} \times \mathbb{C} \mathbb{P}^{n-1} \) by Lemma 3.1. Thus, as \( n \to \infty \), we have

\[
P_1(t) = \frac{t^2}{1-t^2}.
\]

**d=2.** By Lemma 5.1 when \( d > 1 \) and \( n \to \infty \), we have that for every \( i \),

\[
H_c^i(\text{Red}_d(\mathbb{C})) \cong H_c^{i+1}(\text{Irr}_d(\mathbb{C})). \tag{7.2}
\]

For \( V \) a graded vector space, we use \( s^kV \) to denote \( V \) with grading shifted by \( k \) \( (a.k.a \) the \( k \)-th suspension of \( V \) where \( (s^kV)_i = V_{i-k} \). When \( d = 2 \), we have

\[
H_c^*(\text{Red}_2(\mathbb{C})) = H_c^*(\text{Sym}^2(\text{Irr}_1(\mathbb{C})))
\]

\[
\cong \text{Sym}^2(H_c^*(\text{Irr}_1(\mathbb{C})))
\]

\[
\cong s^4\text{Sym}^2H^*(\mathbb{C} \mathbb{P}^\infty)
\]

\[
\cong s^4\mathbb{Q}[e_1, e_2] \quad \text{where } |e_1| = 2, |e_2| = 4
\]

The last isomorphism comes from the fundamental theorem of symmetric polynomials. Thus, we conclude

\[
P_2(t) = \frac{t^5}{(1-t^2)(1-t^4)} \tag{7.3}
\]

**d=3.** There are two nontrivial partition of \( d = 3 \), namely \( 3 = 1 + 1 + 1 \) and \( 3 = 1 + 2 \). Let \( T_{1+1+1} \) denote the stratum corresponding to the partition \( 1 + 1 + 1 = 3 \), and so on. We have \( \text{Red}_3(\mathbb{C}) = T_{1+2} \cup T_{1+1+1} \) where \( T_{1+1+1} \) is closed. The associated long exact sequence gives the following connecting homomorphism:

\[
\delta_i : H_c^i(\text{Sym}^3\text{Irr}_1(\mathbb{C})) \to H_c^{i+1}(\text{Irr}_2(\mathbb{C}) \times \text{Irr}_1(\mathbb{C})).
\]

We now show that the differential \( \delta_i \) must be injective for every \( i \). There is a surjective map \( \text{Poly}_2(\mathbb{C}) \times \text{Irr}_1(\mathbb{C}) \to \text{Red}_3(\mathbb{C}) \), given by the multiplication of two polynomials. The preimage of the closed subspace \( T_{1+1+1} \) is \( T_{1+1} \times T_1 \), while the preimage of the open subspace \( T_{1+2} \) is \( T_2 \times T_1 \). We obtain the following commutative diagram:

\[
\begin{array}{ccc}
H_c^i(\text{Sym}^3\text{Irr}_1(\mathbb{C})) & \xrightarrow{\delta_i} & H_c^{i+1}(\text{Irr}_2(\mathbb{C}) \times \text{Irr}_1(\mathbb{C})) \\
\text{transfer} & & \text{transfer} \\
H_c^i(\text{Sym}^3\text{Irr}_1(\mathbb{C}) \times \text{Irr}_1(\mathbb{C})) & \xrightarrow{\delta_i \otimes \text{id}} & H_c^{i+1}(\text{Irr}_2(\mathbb{C}) \times \text{Irr}_1(\mathbb{C}))
\end{array} \tag{7.4}
\]

The vertical map is a transfer homomorphism, given by including the \( S_3 \)-invariant subspace of \( H_c^i(\text{Irr}_1(\mathbb{C})) \otimes \mathbb{C}^3 \) into the \( S_2 \times S_1 \)-invariant subspace. The differential \( \delta_i : H_c^i(\text{Sym}^3\text{Irr}_1(\mathbb{C})) \to H_c^{i+1}(\text{Irr}_2(\mathbb{C})) \) is an isomorphism for all \( i \) by (7.2). Hence, \( \delta \) must be injective, which implies

\[
H_c^{i+1}(\text{Red}_3(\mathbb{C})) \cong \text{coker}(\delta_i)
\]

\[
\cong S_2 \times S_1\text{-invariant subspace of } H_c^*(\text{Irr}_1(\mathbb{C})) \otimes \mathbb{C}^3
\]

\[
\cong S_3\text{-invariant subspace of } H_c^*(\text{Irr}_1(\mathbb{C})) \otimes \mathbb{C}^3
\]
We calculate the Poincaré series of the numerator and the denominator in the same way as in (7.3), taking their difference and multiply by an appropriate power of $t$ to account for the degree shift, and obtain

$$P_3(t) = \frac{t^{10}}{(1-t^2)(1-t^6)}.$$

$d=4$. A calculation of $P_4(t)$ is already too complex for us to sketch here in any reasonable length. Instead, we will be content with finding the first nonzero stable cohomology. We will show that $b_i(4) = 0$ for any $i < 11$ and that $b_{11}(4) = 1$. Hence, $P_4(t) = t^{11} + O(t^{12})$.

There are four nontrivial partitions of 4. The partition poset is ordered below:

```
1 + 3
  \
2 + 2
  \
1 + 1 + 2
  \
1 + 1 + 1 + 1
```

The three levels of the partition lattice above induce a spectral sequence (3.5) with three columns. All terms in the spectral sequence with total degree $\leq 8$ must be zero, by our previous computations for $d \leq 3$. Below we draw the region of the spectral sequence with total degree $\leq 10$.

By the same argument as in (7.4), the two differentials from the first column $p = 0$ to the second column $p = 1$ in the diagram above must be injective. Consequently, the differential $\delta : E^{1,8}_1 \to E^{2,8}_1$ must be zero. Hence, $E^{2,8}_1$, circled in the diagram, must survive in the $E_{\infty}$-page, contributing to $H^c_{10}(\text{Red}_4(C)) \cong H^c_{11}(\text{Irr}_4(C)) \cong \mathbb{Q}$.

References