TOPIC PROPOSAL
THE MAPPING CLASS GROUP AND THE MODULI SPACE OF RIEMANN SURFACES

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AS DISCUSSED WITH PROF. BENSON FARB

0. Introduction

Closed surfaces are completely classified by Euler characteristic. The next natural step is to study maps between surfaces. Two important classes of maps to be studied are homeomorphisms and finite covers.

The mapping class group $\text{Mod}(S_g)$ of a surface $S_g$ of genus $g$ is defined to be the group of (orientation-preserving) self-homeomorphisms on $S$ up to isotopy. For $S_1 = \text{the torus}$, $\text{Mod}(S_1)$ is just the classical modular group $\text{SL}(2, \mathbb{Z})$, which we can study using linear algebra. Remarkably, many familiar properties of $\text{SL}(2, \mathbb{Z})$ that are proved using linear algebra have natural analogs on $\text{Mod}(S_g)$ for higher genus $g$. In fact, many results about $\text{SL}(2, \mathbb{Z})$ are actually special cases of general ones about $\text{Mod}(S_g)$. The following table highlights just a few examples. In this topic proposal, I will explain the proofs for most of the results listed in the right column of the table below. Both the statements and the ideas in the proofs should be reminiscent of the classical genus 1 case.

<table>
<thead>
<tr>
<th>$\text{SL}(2, \mathbb{Z})$</th>
<th>$\text{Mod}(S_g)$</th>
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<tbody>
<tr>
<td>acts on integer vectors</td>
<td>acts on simple closed curves</td>
</tr>
<tr>
<td>generated by finitely many elementary matrices</td>
<td>generated by finitely many Dehn twists</td>
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<tr>
<td>finitely presentable word problem is solvable by matrix multiplication</td>
<td>finitely presentable word problem is solvable by Alexander’s method</td>
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<tr>
<td>has a finite-index subgroup that is torsion-free</td>
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<tr>
<td>$\text{SL}^\pm(2, \mathbb{Z}) \cong \text{Aut}(\mathbb{Z} \oplus \mathbb{Z})$</td>
<td>$\text{Mod}(S_g)^\pm \cong \text{Out}(\pi_1(S_g))$</td>
</tr>
<tr>
<td>$\text{SL}(2, \mathbb{Z})$ acts on $\mathbb{H} \approx \text{Teich}(S_1)$</td>
<td>$\text{Mod}(S_g)$ acts on $\text{Teich}(S_g)$</td>
</tr>
<tr>
<td>- via isometries in the Poincaré metric</td>
<td>- via isometries in the Teichmüller metric</td>
</tr>
<tr>
<td>- properly discontinuously</td>
<td>- properly discontinuously</td>
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<tr>
<td>- Mahler’s compactness criterion</td>
<td>- Mumford’s compactness criterion</td>
</tr>
<tr>
<td>- $\mathbb{H}/\text{SL}(2, \mathbb{Z})$ is finitely covered by an aspherical manifold</td>
<td>- $\text{Teich}(S_g)/\text{Mod}(S_g)$ is finitely covered by an aspherical manifold</td>
</tr>
<tr>
<td>each element is one of the three types: elliptic, parabolic, hyperbolic (or Anosov)</td>
<td>each element is one of the three types: periodic, reducible, pseudo-Anosov</td>
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<tr>
<td>Jordan’s canonical form</td>
<td>Thurston’s canonical form</td>
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To study finite cover of surfaces, we first consider the action of the deck transformations on the first homology group of the cover. In the last section, I will briefly discuss a result of Chevalley-Weil that generalizes the Riemann-Hurwitz formula.

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1. The Mapping Class Group

1.1. Finiteness Properties. Let \( S := S^{k}_{g,n} \) be a surface of genus \( g \), with \( k \) boundary components and \( n \) punctures.

**Definition 1.** The mapping class group of \( S \) is

\[
\text{Mod}(S) := \pi_0(\text{Homeo}^+(S;\partial S))
\]

where \( \text{Homeo}^+(S;\partial S) \) is the group of orientation-preserving self-homeomorphisms on \( S \) that fix boundary pointwise.

**Example 1** (Alexander’s lemma). \( \text{Mod}(D^2) = 1 \).

**Example 2.** \( \text{Mod}(A) = \mathbb{Z} \). The generator is a Dehn twist.

Given \( \text{Mod}(S_g) \) as an abstract group, a natural question to ask is whether the group is finitely generated or even finitely presentable.

**Theorem 1.1** (Dehn-Lickorish). For \( g \geq 0 \), \( \text{Mod}(S_g) \) is generated by finitely many Dehn twists about nonseparating simple closed curves.

**Proof.** (sketch). We prove a stronger claim that the pure mapping class group \( P\text{Mod}(S_{g,n}), g \geq 2 \) is generated by finitely many Dehn twists. We induct on \( n \) using the Birman exact sequence. We induct on \( g \) by studying the action of \( P\text{Mod}(S_{g,n}) \) on the complex of nonseparating curves \( \tilde{N}(S_{g,n}) \) (connected when \( g \geq 2 \)) which is transitive on vertices and edges. The induction hypothesis exactly applies to the edge stabilizers.

The proof of finite presentation uses an analogous idea.

**Theorem 1.2.** \( \text{Mod}(S) \) is finitely presented.

**Proof.** (sketch). By “capping a boundary component”, it suffices to consider surfaces with nonempty boundary. The vertices of the arc complex \( \mathcal{A}(S) \) are free isotopy classes of essential simple proper arcs, and two vertices are connected by an edge if they have disjoint representatives. \( \mathcal{A}(S) \) is contractible by a construction of an explicit flow onto the star of a vertex. \( \text{Mod}(S) \) acts cocompactly on \( \mathcal{A}(S) \). Each vertex stabilizer is finitely presented, and each edge stabilizer is finitely generated. It is a general fact that any group admitting such action is finitely presented.

It is possible to write down explicit presentations for \( \text{Mod}(S) \). One presentation that I will use in the next section is Wajnryb’s presentation.

1.2. Surface Bundles and the Mapping Class Groups. A surface bundle is a fiber bundle over a base space \( B \) with fiber \( S_g \). In this section, I will explain how our knowledge about mapping class groups can be transferred into knowledge about surface bundles, and vice versa.

Analogous to vector bundles, all surface bundle are pullbacks from a universal \( S_g \)-bundle. More precisely, fixing a base space \( B \), there is a bijective correspondence given by bundle pullbacks:

\[
\begin{align*}
\{ & \text{Isomorphism classes of oriented } S_g \text{-bundles over } B \} \\
\{ & \text{Homotopy classes of maps } B \rightarrow B\text{Homeo}^+(S_g) \}
\end{align*}
\]

Furthermore, for \( g \geq 2 \), \( \text{Homeo}_0(S_g) \) is contractible. Thus, the classifying space \( B\text{Homeo}^+(S_g) \) is a \( K(\text{Mod}(S_g),1) \) space. The correspondence above can be further reduced to

\[
\begin{align*}
\{ & \text{Isomorphism classes of oriented } S_g \text{-bundles over } B \} \\
\{ & \text{Conjugacy classes of representations } \pi_1(B) \rightarrow \text{Mod}(S_g) \}
\end{align*}
\]

Moreover, since \( H^*(B\text{Homeo}^+(S_g);\mathbb{Z}) \cong H^*(\text{Mod}(S_g);\mathbb{Z}) \), the cohomology of \( \text{Mod}(S_g) \) contains all the characteristic classes of \( S_g \)-bundles. This connection provides both motivation and tools for studying the homology/cohomology of \( \text{Mod}(S_g) \).

**Proposition 1.3.** For \( g \geq 3 \), \( H_1(\text{Mod}(S_g);\mathbb{Z}) = 0 \).

**Proof.** (sketch). (Harer) For any genus \( g \), the abelianization of \( \text{Mod}(S_g) \) is generated by the image of a single Dehn twist according to Theorem 1.1 and the change of basis principle. For \( g \geq 3 \), \( S_g \) contains a lantern. The lantern relation kills the generator after abelianization. \( \square \)
The proof of next theorem illustrates how we can combine the perspectives coming from mapping class groups and from surface bundles to obtain a complete picture for both.

**Theorem 1.4** (Harer). For \( g \geq 4 \), \( H_2(\text{Mod}(S_g); \mathbb{Z}) \cong \mathbb{Z} \)

**Proof. (sketch).** On the one hand, applying Hopf formula to the Wajnryb’s presentation of \( \text{Mod}(S_g) \) shows that \( H_2(\text{Mod}(S_g); \mathbb{Z}) \) is cyclic. On the other hand, the Meyer signature cocycle is a characteristic class of \( S_g \)-bundle in dimension 2, or equivalently, an (nonzero if \( g \geq 4 \)) element in \( H^2(\text{Mod}(S_g); \mathbb{Z}) \). By the universal coefficient theorem, \( H_2(\text{Mod}(S_g); \mathbb{Z}) \) must have rank at least 1, so must be \( \mathbb{Z} \).

As a corollary, signature is the only 2-dimensional characteristic class of \( S_g \)-bundle \((g \geq 4)\) up to multiples.

1.3. The Symplectic Representation. \( \text{Mod}(S_g) \) acts on \( H_1(S_g; \mathbb{Z}) \cong \mathbb{Z}^{2g} \) preserving a symplectic form defined by the algebraic intersection number. So we have a representation

\[
\Psi : \text{Mod}(S_g) \to \text{Sp}(2g, \mathbb{Z}).
\]

\( \Psi \) can be seen as a linear approximation of \( \text{Mod}(S_g) \). Its kernel, called the Torelli group \( \mathcal{I}(S_g) \), can be viewed as the “nonlinear” part of \( \text{Mod}(S_g) \).

Given an integer \( m \geq 2 \), define the level \( m \) congruence subgroup \( \text{Mod}(S_g)[m] \) to be the kernel of the composition

\[
\text{Mod}(S_g) \xrightarrow{\Psi} \text{Sp}(2g, \mathbb{Z}) \xrightarrow{\text{mod } m} \text{Sp}(2g, \mathbb{Z}/m\mathbb{Z})
\]

**Theorem 1.5.** Let \( g \geq 1 \), \( \mathcal{I}(S_g) \) is torsion free. More generally, \( \text{Mod}(S_g)[m] \) is a torsion-free finite-index subgroup of \( \text{Mod}(S_g) \) for \( m \geq 3 \).

**Proof. (sketch).** The statements are obviously true for \( g = 1 \). Given any finite order \( f \in \mathcal{I}(S_g) \), by the Lefschetz formula, the Lefschetz number of \( f \) is \( \chi(S_g) < 0 \). On the other hand, we can lift \( f \) to a finite order homeomorphism \( \phi \) by Theorem 1.3. Average a Riemannian metric on \( S_g \) so that \( \phi \) becomes an isometry. Then \( \phi \) is infinitesimally a rotation at each fixed point, and thus its Lefschetz number must be nonnegative. This is a contradiction. The second statement follows from the first and the fact that \( \text{Sp}(2g, \mathbb{Z})[m] \) is torsion free for \( g \geq 3 \).

From the group extension

\[
1 \to \mathcal{I}(S_g) \to \text{Mod}(S_g) \xrightarrow{\Psi} \text{Sp}(2g, \mathbb{Z}) \to 1
\]

we obtain a representation

\[
\rho : \text{Sp}(2g; \mathbb{Z}) \to \text{Out}(\mathcal{I}(S_g))
\]

Therefore, the abelian group \( H^*(\mathcal{I}(S_g); \mathbb{Z}) \) is an \( \text{Sp}(2g; \mathbb{Z}) \)-module, allowing us to apply the representation theory of symplectic group to study \( H^*(\mathcal{I}(S_g); \mathbb{Z}) \).

Let \( \mathcal{K}(S_g) \) be the subgroup of \( \mathcal{I}(S_g) \) generated by the Dehn twists about separating simple closed curves. In 1970s, Birman asked whether \( \mathcal{K}(S_g) = \mathcal{I}(S_g) \). The following beautiful theorem gives the negative answer to Birman’s question.

**Theorem 1.6** (Johnson). For \( g \geq 2 \), there is a surjective homomorphism \( \tau : \mathcal{I}(S_{g,1}) \to \wedge^3 H_1(S_{g,1}; \mathbb{Z}) \) where \( \mathcal{K}(S_{g,1}) \) is contained in the kernel. In particular, \( \mathcal{K}(S_{g,1}) \) has infinite index in \( \mathcal{I}(S_{g,1}) \).

**Proof. (sketch).** We will prove a slightly weaker statement in \( \mathbb{Q} \)-coefficients. \( \tau \) is constructed by looking at the action of \( \mathcal{I}(S_{g,1}) \) on the first term in the lower central series of the free group \( \pi_1(S_{g,1}) \cong F_{2g} \). \( \tau \) factors through an \( \text{Sp}(2g; \mathbb{Q}) \)-equivariant map of abelian groups \( H_1(\mathcal{I}(S_{g,1}); \mathbb{Q}) \to \wedge^3 H \otimes \mathbb{Q} \). Decompose \( \wedge^3 H \otimes \mathbb{Q} \) into \( \text{Sp}(2g; \mathbb{Q}) \)-irreps: \( \wedge^3 H \cong (\wedge^3 H/\mathbb{H}) \oplus H \). By explicit computations, we see that \( \mathcal{K}(S_{g,1}) \) maps to zero and we also find elements in \( \mathcal{I}(S_{g,1}) \) which maps to nontrivial elements in each of the two irreps. Then by Schur’s lemma, \( \tau \) must be surjective.
1.4. An Algebraic Description. The following remarkable theorem gives a purely algebraic description of the (extended) mapping class groups $\text{Mod}^{±}(S_g) := \pi_0(\text{Homeo}^{±}(S_g))$. In addition to the somewhat surprising statement of the theorem, its proof uses crucial observations from geometric group theory and hyperbolic geometry.

**Theorem 1.7** (Dehn-Nielsen-Baer). For $g \geq 1$, $\sigma : \text{Mod}^{±}(S_g) \to \text{Out}(\pi_1(S_g))$ is an isomorphism.

*Proof. (sketch).* When $g = 1$, both sides equal to $\text{SL}^{±}(2, \mathbb{Z})$. Assume $g \geq 2$. $\sigma$ is injective because $S_g$ is a $K(\pi, 1)$. For surjectivity, any automorphism of $\pi_1(S_g)$ is a quasi-isometry on word metric, inducing a quasi-isometry on $\mathbb{H}$ on which $\pi_1$ acts cocompactly. A quasi-isometry on $\mathbb{H}$ induces a homeomorphism on $\partial \mathbb{H}$ which preserving the linkings at infinity. Using the linkings, we can construct a mapping class that projects to the given outer automorphism. 

2. The Teichmüller Space and the Nielsen-Thurston classification

2.1. The Teichmüller Space and Its Incarnations. Fix a topological surface $S$ with negative Euler characteristics. A marking of hyperbolic surface is a pair $(X, \phi)$ where $X$ is a hyperbolic surface with geodesic boundary components and $\phi : S \to X$ is a homeomorphism. We define the Teichmüller space $\text{Teich}(S)$ to be the set of marked hyperbolic surfaces homeomorphic to $S$ up to isotopy.

For $g \geq 2$, the uniformization theorem gives the following bijective correspondences

$$
\text{Teich}(S_g) := \{\text{marked hyperbolic surfaces}\}/\text{isotopy} \quad (1)
$$

$$
\longleftrightarrow \{\text{marked Riemann surfaces}\}/\text{isotopy} \quad (2)
$$

$$
\longleftrightarrow \{\text{discrete faithful } \pi_1(S_g) \to PSL(2; \mathbb{R})\}/PGL(2; \mathbb{R}) \quad (3)
$$

We equip $\text{Teich}(S_g)$ with the obvious topology coming from (3).

2.2. The Topology of the Teichmüller Space. The Teichmüller space is defined to be the quotient of a huge space (the space of all markings) by the action of an infinite dimensional group $\text{Diff}_0(S_g)$. Remarkably, the quotient $\text{Teich}(S_g)$ turns out to be finite dimensional. Moreover, it is contractible.

**Theorem 2.1.** For $g \geq 2$, there is a homeomorphism

$$
FN : \text{Teich}(S_g) \longrightarrow \mathbb{R}^{3g-3} \times \mathbb{R}^{3g-3}.
$$

*Proof. (sketch).* (Fenchel-Nielsen) A hyperbolic hexagon is determined up to isometry by the lengths of its three alternating sides. Hence, an hyperbolic pair of pants is determined up to isometry by the lengths of its three boundary components. A marked hyperbolic surface of genus $g$ can be decomposed by $3g-3$ disjoint simple closed curves into pairs of pants. We need extra $3g-3$ twisting parameters to record how pants are glued together. 

2.3. The Geometry of the Teichmüller Space. Teichmüller’s theorem solves the extremal problem of dilatations of quasiconformal homeomorphism. This allows us to define a metric on $\text{Teich}(S_g)$.

Let $X, Y$ be Riemann surfaces and let $q_x, q_y$ be holomorphic quadratic differentials on $X$ and $Y$ respectively. A Teichmüller mapping from $X$ to $Y$ is a homeomorphism $f : X \to Y$ such that $f$ maps the zeros of $q_x$ to the zeros of $q_y$, and $\forall p \in X$ where $q_x$ is nonzero, under the natural coordinates of $q_x, q_y$ centered at $p, f(p)$, $f$ can be written as

$$
f(x + iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y
$$

for some $K \in \mathbb{R}_{>0}$.

**Theorem 2.2** (Teichmüller’s existence and uniqueness theorem). Given any homeomorphism $f : X \to Y$ between closed Riemann surfaces of genus $g \geq 1$, there exists a Teichmüller mapping $h : X \to Y$ that is homotopic to $f$. Furthermore, if the given $f$ is a quasiconformal map, then the dilatations satisfy $K_f \geq K_h$ with the equality holds if and only if $f \circ h^{-1}$ is conformal.
Definition 2. For $g \geq 1$, given any $X = [(x, \phi)], Y = [(y, \psi)] \in Teich(S_g)$, the Teichmüller distance is defined to be

$$d_{Teich}(X, Y) := \frac{1}{2} \log K_h$$

where $h : X \to Y$ is a Teichmüller mapping homotopic to the change-of-marking map $\psi \circ \phi^{-1} : X \to Y$.

Remark 1. For $g \geq 2$, the Teichmüller mapping $h$ is unique. For $g = 1$, the Teichmüller mapping is unique up to composition with a conformal map which does not change the dilatation $K_h$. Hence, $d_{Teich}$ is well-defined. $(Teich(S_g), d_{Teich})$ is a complete metric space. The bi-infinite geodesics are precisely the Teichmüller lines.

2.4. The Action of the Mapping Class Groups on the Teichmüller space. $Mod^\pm(S_g)$ acts on $Teich(S_g)$ by changing the markings. The action is by isometries with respect to $d_{Teich}$.

In fact, by the Dehn-Nielsen-Baer theorem and the bijection in Section 2.1, the following are just different incarnations of one action:

$$\text{Out}(\pi_1(S_g)) \simeq \text{Mod}^+(S_g)/\text{Isom}(\mathbb{H})$$

Definition 3. The moduli space of Riemann surfaces of genus $g$ is the quotient space

$$\mathcal{M}(S_g) := Teich(S_g)/Mod(S_g).$$

Theorem 2.3 (Fricke). For $g \geq 1$, the action of $Mod(S_g)$ on $Teich(S_g)$ is properly discontinuous.

Proof. (sketch). Suppose $B$ is a compact subset of $Teich(S_g)$ and $f \in Mod(S_g)$ satisfis $\exists \mathcal{X} \in (f \cdot B) \cap B$. Then $d_{Teich}(X, f \cdot X)$ is bounded from above. Hence, $f$ stretches the lengths of closed geodesics on $X$ by a bounded amount. By the discreteness of the length spectrum on $\mathcal{X}$, there are only finitely many closed geodesics with length bounded from above. By Alexander’s lemma, the actions on closed geodesics determines $f$ up to finitely many choices.

Example 3 (genus 1). $Mod(S_1) \cong SL_2(\mathbb{Z})$. $Teich(S_1) \cong \mathbb{H}$. $\mathcal{M}(S_1) = \mathbb{H}/SL(2, \mathbb{Z})$ is the modular curve. The two cone points corresponds to the square and the hexagonal tort. The cusp corresponds an “infinitely thin” torus.

Proper discontinuity of the action implies that $\mathcal{M}(S_g)$ inherits a metric from $Teich(S_g)$. By Theorem 1.5, $Mod(S_g)$ has a finite-index torsion-free subgroup.

Corollary 2.4. For $g \geq 1$, $\mathcal{M}(S_g)$ is an orbifold finitely covered by an aspherical manifold.

The action of $Mod(S_g)$ on $Teich(S_g)$ is never cocompact because we can leave every compact subset of $\mathcal{M}(S_g)$ by pinching a simple closed curve. The next theorem shows that this is in fact the only way to escape to infinity in $\mathcal{M}(S_g)$. Let $\mathcal{M}_r(S_g)$ denote the subset of $\mathcal{M}(S_g)$ consisting of Riemann surfaces whose shortest closed geodesics have length $\geq \epsilon$.

Theorem 2.5 (Mumford). For $g \geq 1$, $\mathcal{M}_r(S_g)$ is compact for any $\epsilon > 0$.

Proof. (sketch). The case $g = 1$ follows from Mahler’s compactness criterion. If $g \geq 2$, using Gauss-Bonnet, we can find a constant $L = L(g)$ such that every hyperbolic surface of genus $g$ admits a pants decomposition $\{\gamma_i\}_{i=1}^{3g-3}$ such that $\text{length}(\gamma_i) \leq L, \forall i$.

Since there are finitely many topological type of pants decompositions of $S_g$, given any infinite sequence $\{\mathcal{X}_i\}_{i=0}^{\infty} \in \mathcal{M}_r(S_g)$, we can assume (by passing to an infinite subsequence) that there is a pants decomposition with lengths no greater than $L$ on each $\mathcal{X}_i$. We can lift $\mathcal{X}_i$ to $\mathcal{X}_\tilde{i} \in Teich(S_g)$ such that, in the Fenchel-Nielsen coordinates for this specific pants decomposition, the length parameters are contained in $[\epsilon, L]$, while the twisting parameters are contained in $[0, 2\pi]$. Hence, the sequence must contain a convergent subsequence.

2.5. The Nielsen-Thurston Classification of Mapping Classes.

Definition 4. Given a mapping class $f \in Mod(S_g), g \geq 2$.

1. $f$ is periodic if $f$ has finite order.

2. $f$ is reducible if there is a nonempty collection $\{c_1, \ldots, c_m\}$ of isotopy classes of essential simple closed curves on $S_g$ (called a reduction system of $f$) so that $i(c_i, c_j) = 0, \forall i, j$ and $\{f(c_i)\} = \{c_i\}$. 

(3) \( f \) is pseudo-Anosov if \( f = [\phi] \) where \( \phi \) is a pseudo-Anosov homeomorphism. A homeomorphism \( \phi : S_g \to S_g \) is pseudo-Anosov if there is a pair of transverse measured foliation \((\mathcal{F}^u, \mu_u), (\mathcal{F}^s, \mu_s)\) on \( S_g \) and a real number \( \lambda > 1 \) such that
\[
\phi \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda \mu_u), \quad \phi \cdot (\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1} \mu_s)
\]

Nielsen’s realization theorem characterizes finite order mapping classes in terms of finite order homeomorphisms.

**Theorem 2.6** (Nielsen’s realization theorem, the cyclic case). For \( g \geq 1 \), if \( f \in \text{Mod}(S_g) \) has finite order \( k \), then there exists a homeomorphism \( \phi : S_g \to S_g \) such that \( f = [\phi] \) and \( \phi^k = \text{id} \).

**Proof.** (sketch). We induct on the number of prime divisors of the order \( k \). We know that \( \langle f \rangle \) cannot act freely on \( \text{Teich}(S_g) \) otherwise the quotient would be a finite dimensional \( K(Z/kZ, 1) \), so some power of \( f \) has fixed point in \( \text{Teich}(S_g) \). If \( k \) is a prime, we are done. Otherwise, say \( k = p \cdot l \) where \( p \) is a prime. The induction hypothesis implies that \( \text{Fix}(f^p) \) is nonempty and contractible. \( \langle f^l \rangle \) acts on \( \text{Fix}(f^p) \). The action cannot be free as above. Since \( p \) is a prime, \( f \) has a fixed point.

The following theorem is a cornerstone of the theory of mapping class groups. It gives a concrete representation of any single mapping class in \( \text{Mod}(S_g) \), in the same way that the Jordan canonical form gives a concrete representation of any matrix.

**Theorem 2.7** (Nielsen-Thurston). If a mapping class \( f \in \text{Mod}(S_g) \) \((g \geq 2)\) is neither periodic nor reducible, then \( f \) is pseudo-Anosov.

**Proof.** (sketch). We define the translation length \( \tau(f) := \inf\{d_{\text{Teich}}(\mathcal{X}, f \cdot \mathcal{X}) \mid \mathcal{X} \in \text{Teich}(S_g)\} \). Consider three possibilities: (1) the inf is achieved and \( \tau(f) = 0 \); (2) \( \tau(f) \) is not achieved; (3) the inf is achieved and \( \tau(f) > 0 \).

In case (1), \( f \) fixes some \( \mathcal{X} \in \text{Teich}(S_g) \) so corresponds to a finite order isometry. Thus \( f \) is periodic. In case (2), we can construct a reduction system which \( f \) preserves using Mumford’s compactness criterion and Wolpert’s lemma. In case (3), since \( \text{Teich}(S_g) \) is a complete geodesic metric space, \( f \) must preserve a geodesic and acts as Teichmüller maps for points along the geodesics with the initial and the terminal quadratic differentials being equal. Such means that \( f \) is a pseudo-Anosov map on \( S_g \).

**Remark 2.** The converse of Theorem 2.7 is also true: a pseudo-Anosov mapping class is neither periodic nor reducible. To see this, the stable and unstable foliations induce a singular Euclidean metric on \( S \). For any simple closed curve \( \alpha \) on \( S \), the length of \( f^m(\alpha) \) goes to infinity under such metric. This phenomenon cannot happen if \( f \) were periodic or reducible. However, it is possible for a mapping class to be both periodic and reducible.

Theorem 2.7 allows us to write a mapping class in a canonical form. Given any mapping class \( f \in \text{Mod}(S) \), after passing to some power \( f^k \), we can decompose \( S \) into subsurfaces so that \( f^k \) is either identity or a Dehn twist or pseudo-Anosov on each individual subsurface.

### 3. Finite Covers of Surfaces

After homeomorphisms, the next class of maps that we would like to study is finite covers (regular or branched) of surfaces. This class of maps is large enough to arise ubiquitously in mathematics (e.g. every holomorphic map between compact Riemann surfaces is a branched cover) while special enough to admit an interesting theory.

Suppose \( p : X \to Y \) is a branched cover of closed oriented surfaces with a finite deck group \( G := \text{Gal}(X/Y) \). \( B \subset Y \) are the branch points. Assume further \( Y \) has positive genus.

**Theorem 3.1** (Chevalley-Weil). The following is an isomorphism of \( G \)-modules.
\[
H_1(X; \mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q} \oplus (\mathbb{Q}[G])^{-\chi(Y)} \oplus \bigoplus_{b \in B} \frac{\mathbb{Q}[G]}{\mathbb{Q}[G/G_b]}
\]

where the first two 1-dimensional representations are trivial. \( G_b \) stands for the stabilizer of \( G \) at \( b \in X \) where \( p(b) = b \).
Proof. (sketch). [Proof I] We generalize the proof of Riemann-Hurwitz. Let \((Y; V, E, F)\) be a triangulation of \(Y\) so that vertex set \(V\) includes all branched points. Lift \((V, E, F)\) to a triangulation of \(X\), \((X; \tilde{V}, \tilde{E}, \tilde{F})\). We obtain a \(G\)-equivariant chain complex for \(X\)

\[
0 \xrightarrow{\partial_3} \mathbb{Q}[\tilde{F}] \xrightarrow{\partial_2} \mathbb{Q}[\tilde{E}] \xrightarrow{\partial_1} \mathbb{Q}[\tilde{V}] \xrightarrow{\partial_0} 0.
\]
Computations in the Grothendieck ring of finite dimensional \(G\)-modules yield the desired formula.

[Proof II] We compute the characters using the Riemann-Hurwitz and the Lefschetz formulae, and check that both sides are equal. Hence the two \(G\)-representations must be isomorphic since \(\mathbb{Q}\) has characteristic zero and \(G\) is a finite group. \(\square\)

Remark 3. Since we establish the isomorphism by computing characters, the isomorphism is not canonical. However, in some simple cases (e.g. when \(G\) is cyclic), we can obtain the isomorphism by an explicit topological construction.

References


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