This exam contains 10 pages (including this cover page) and 9 problems. Check to see if any pages are missing and print your name in the space at the top of this page.

The only resources you may use are a calculator (not a cell phone), and your notecard.

You must show and explain your work in a reasonably neat and coherent way to be able to receive full credit. All answers should be simplified.

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1. Telephone calls enter a college switchboard at an average rate of two calls every 3 minutes.

(a) (10 points) What distribution would you use to model the number of calls arriving in a 9-minute period?

\[ \lambda = 2 \times 3 = 6 = \text{average \# of calls in 9 minutes} \]

Poisson (6)

\[ P(k) = e^{-\lambda} \frac{\lambda^k}{k!} \]

(b) (15 points) What is the probability of five or more calls arriving in a 9-minute period?

\[ 1 - \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = 0.7149 \]
2. (30 points) Kristina and Danielle are repeatedly playing a game. Each time they play the game, Kristina has probability $\frac{1}{3}$ of winning. The first person to win 5 games wins the match. What is the probability that Danielle will win if she currently leads by a score of 2 games to 1?

\[
P(D \text{ wins } 5-1) + P(D \text{ wins } 5-2) + P(D \text{ wins } 5-3) + P(D \text{ wins } 5-4)\]

\[
\left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \left(\frac{1}{3}\right) \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^2 \left(\frac{2}{3}\right)\]

\[
\frac{8}{27} + \frac{8}{81} \cdot \frac{4}{3} + \frac{8 \cdot 4 \cdot 2}{243 \cdot 2} + \frac{8 \cdot 4 \cdot 2 \cdot 2}{243 \cdot 3 \cdot 2} \]

\[
\frac{432}{1458} + \frac{432}{1458} + \frac{288}{1458} + \frac{1440}{1458} \]

\[
\frac{1312}{1458} \]

\[
0.8999
\]
3. Suppose we roll two tetrahedral dice that have 1, 2, 3, and 4 on their four side. Let $X$ be the sum of the values.

(a) (5 points) What is the sample space for this experiment?

$$\mathcal{S} = \{2, 3, 4, 5, 6, 7, 8\}$$

(b) (5 points) What is the distribution of $X$?

- $P(X=2) = \frac{1}{16}$
- $P(X=3) = \frac{2}{16}$
- $P(X=4) = \frac{3}{16}$
- $P(X=5) = \frac{4}{16}$
- $P(X=6) = \frac{5}{16}$
- $P(X=7) = \frac{6}{16}$
- $P(X=8) = \frac{1}{16}$

(c) (5 points) What is the expected value of $X$?

$$E(X) = 2\left(\frac{1}{16}\right) + 3\left(\frac{2}{16}\right) + 4\left(\frac{3}{16}\right) + 5\left(\frac{4}{16}\right) + 6\left(\frac{5}{16}\right) + 7\left(\frac{6}{16}\right) + 8\left(\frac{1}{16}\right)$$

$$= \frac{2 + 6 + 12 + 20 + 30 + 42 + 8}{16}$$

$$= \frac{80}{16}$$

$$= 5$$

(d) (5 points) What is the variance of $X$?

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= 4\left(\frac{1}{16}\right) + 9\left(\frac{2}{16}\right) + 16\left(\frac{3}{16}\right) + 25\left(\frac{4}{16}\right) + 36\left(\frac{5}{16}\right) + 49\left(\frac{6}{16}\right) + 64\left(\frac{1}{16}\right) - 5^2$$

$$= \frac{4 + 18 + 48 + 100 + 108 + 98 + 64}{16} - 25$$

$$= \frac{4}{16}$$

$$= \frac{40}{16}$$

$$= \frac{5}{2}$$
4. (25 points) Suppose we draw 2 cards out of a deck of 52 without replacement. Let $A$ be the event that the first card is an ace and $B$ be the probability that the second card is a spade. Are these events independent?

\[
\begin{align*}
P(A) &= \frac{4}{52} = \frac{1}{13} \\
P(B) &= \left(\frac{13}{52}\right)\left(\frac{12}{51}\right) + \left(\frac{39}{52}\right)\left(\frac{13}{51}\right) \\
&= \frac{1}{13} + \frac{13}{41 \cdot 13} \\
&= \frac{14}{41 \cdot 13} \\
&= \frac{1}{4} \\
P(A \cap B) &= \left(\frac{1}{52}\right)\left(\frac{12}{51}\right) + \left(\frac{39}{52}\right)\left(\frac{13}{51}\right) \\
&= \frac{12 + 39}{52 \cdot 51} \\
&= \frac{1}{52} \\
P(A)P(B) &= \frac{1}{13} \cdot \frac{1}{4} = \frac{1}{52}
\end{align*}
\]

They are independent.
5. (25 points) Ten people call an electrician and ask her to come to their house on a randomly chosen day of the work week (Monday through Friday). What is the probability she has at least one day with no jobs?

\[ A_i = \text{no job on day } i \]

\[
P(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) = \sum_{i=1}^{5} P(A_i) - \sum_{i \neq j \in \{1, 2, 3, 4, 5\}} P(A_i \cap A_j) + \sum_{i \neq j \neq k \in \{1, 2, 3, 4, 5\}} P(A_i \cap A_j \cap A_k) - \sum_{i \neq j \neq k \neq l \in \{1, 2, 3, 4, 5\}} P(A_i \cap A_j \cap A_k \cap A_l)
\]

\[= 5 \left( \frac{4}{6} \right)^{10} - \left( \frac{3}{6} \right) \left( \frac{2}{6} \right)^{10} + \left( \frac{5}{6} \right) \left( \frac{2}{6} \right)^{10} - \left( \frac{5}{6} \right) \left( \frac{1}{6} \right)^{10} = 0.41775\]
6. (15 points) The probability of a royal flush in poker is $1/649,740$. How large must $n$ be for the probability of having no royal flush in $n$ hands to be smaller than $1/2$?

\[
\left( \frac{649,739}{649,740} \right)^n < \frac{1}{2}
\]

\[
n > \log_\frac{649,739}{649,740} \frac{1}{2}
\]

\[
n > \log \frac{1}{2} \cdot \frac{649,740}{649,739}
\]

\[
n > \frac{0.30103}{0.00000465}
\]

\[
n > 65,024.103
\]

\[n \text{ must be at least } 45,036.6\]
7. (20 points) An airline finds that 4% of the passengers that make reservations do not show up. Consequently, they sell 100 tickets for a plane that only has 98 seats. Find the probability that every person who shows up for the flight will find a seat available.

\[ P \left( \sum_{k=0}^{100} \binom{100}{k} (0.96)^{100-k} (0.04)^k \right) = 1 \]

0.912837
8. (15 points) Suppose $A$ and $B$ are independent, $A$ and $C$ are independent, and $B$ and $C$ are disjoint ($B \cap C = \emptyset$). Are $A$ and $B \cup C$ independent? If so, prove (including justification for your steps). If not, provide an example where independence does not hold. (Hint: Draw a picture.)

\[
P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))
\]
\[
= P(A \cap B) + P(A \cap C) \quad \text{by } B \cap C = \emptyset
\]
\[
= P(A)P(B) + P(A)P(C) \quad \text{by } A, B \text{ indep., } A, C \text{ indep.}
\]
\[
= P(A)(P(B) + P(C))
\]
\[
= P(A) P(B \cup C) \quad \text{by } B \cap C = \emptyset
\]

$A$ and $B \cap C$ are independent.
9. A house has four bedrooms for 14 students. The front room holds 5 students, the back room holds 3 students, the attic holds 2 students, and the basement holds 4 students. The students are randomly assigned to the bedrooms.

(a) (10 points) Find the number of ways the students can be assigned to the bedrooms. Make sure to explain your reasoning.

\[
\binom{14}{5, 3, 2, 4} = \frac{14!}{5!3!2!4!} = \frac{126485760}{120} = 252680
\]

number of ways to split 14 items into 4 groups of size 5, 3, 2, and 4.

(b) (15 points) If Bryan and Peter have already been assigned to the front room, what is the probability that Eric and Dean are also assigned to the front room? Make sure to explain your reasoning.

\[
\binom{3}{2} = \frac{3!2!1!}{2!1!1!} = \frac{6}{6} = 1
\]

\[
\binom{12}{2} = \frac{12!10!}{2!10!10!} = \frac{12}{2} = 6
\]

\[
\binom{12}{2} \text{ ways left to choose Eric and Deans spots, } \binom{2}{2} \text{ of them have both in the front room.}
\]