Exam 2 Review

Math 4653, Fall 2017

November 2, 2017

1. 

\[
P(\text{first heads } | \text{ at least 1 heads}) = \frac{P(\text{first heads and at least 1 heads})}{P(\text{at least 1 heads})} \\
= \frac{P(\text{first heads})}{P(\text{at least 1 heads})} \\
= \frac{1}{2} \\
= \frac{3}{4} \\
= \frac{2}{3}
\]

2. Let D be the event a randomly chosen family has a dog and C be the event a randomly chosen family has a cat. We know \(P(D) = 0.4\), \(P(C) = 0.3\), and \(P(C | D) = 0.25\).

(a) 

\[
P(C \cup D) = P(C) + P(D) - P(C \cap D) \\
= P(C) + P(D) - P(D)P(C | D) \\
= 0.3 + 0.4 - (0.4)(0.25) \\
= 0.6
\]

(b) 

\[
P(D | C) = \frac{P(C \cap D)}{P(C)} \\
= \frac{P(D)P(C | D)}{P(C)} \\
= \frac{(0.4)(0.25)}{0.3} \\
= \frac{1}{3}
\]
3. 

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \]
\[ = P(A) + P(B) + P(C) - P(A \cap B) - P(A|C)P(C) - P(B|C) + P(A \cap B \cap C) \]
\[ = 0.3 + 0.4 + 0.5 - (0.3)(0.5) - (0.5)(0.1) + 0 \]
\[ = 1 \]

4. 

\[ \left( \frac{1}{36} \right) \left( \frac{35}{36} \right) + \left( \frac{2}{36} \right) \left( \frac{33}{36} \right) + \left( \frac{3}{36} \right) \left( \frac{30}{36} \right) + \left( \frac{4}{36} \right) \left( \frac{26}{36} \right) + \]
\[ \left( \frac{5}{36} \right) \left( \frac{21}{36} \right) + \left( \frac{6}{36} \right) \left( \frac{15}{36} \right) + \left( \frac{5}{36} \right) \left( \frac{10}{36} \right) + \left( \frac{4}{36} \right) \left( \frac{6}{36} \right) + \]
\[ \left( \frac{3}{36} \right) \left( \frac{3}{36} \right) + \left( \frac{2}{36} \right) \left( \frac{1}{36} \right) + \left( \frac{1}{36} \right) \left( \frac{0}{36} \right) = 0.4437 \]

(see picture below)
5. \((0.2)(0.25) + (0.3)(0.5) + (0.5)(1) = 0.7\)

\[ \begin{array}{c}
\text{doesn’t know} \\
0.2 \\
0.25 \quad \text{gets right} \\
0.75 \quad \text{gets wrong} \\
0.3 \quad \text{can narrow down} \\
0.5 \quad \text{gets right} \\
0.5 \quad \text{gets wrong} \\
0.5 \quad \text{gets wrong} \\
0.25 \quad \text{gets right} \\
0.75 \quad \text{gets wrong} \\
1 \quad \text{gets right} \\
0 \quad \text{gets wrong} \\
\end{array} \]

6. \(P(1 \text{ sent} \mid 1 \text{ received}) = \frac{P(1 \text{ sent and received})}{P(1 \text{ received})} = \frac{(0.5)(0.9)}{0.5} = 0.9.\)

7. \[
P(2\text{-headed penny} \mid 4 \text{ heads}) = \frac{P(2\text{-headed penny and 4 heads})}{P(4 \text{ heads})} = \frac{\frac{1}{5}}{\frac{1}{5}} = 1/5
\]
\[
= \frac{4/(5 \times 2^4) + 1/5}{4/(5 \times 2^4) + 1/5} = \frac{4}{5}
\]

8. \(P(X = i, Y = j) = \begin{cases} \frac{1}{12} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases} \)

They are not independent because \(P(X = i, Y = i) = 0\) and \(P(X = i)P(Y = i) = \frac{1}{16}.\)

9. \(P(X = 2 \mid Y = 3) = 0\)

\(P(Y = 3 \mid X = 3) = 0.05/0.35 = 1/7\)
10. \[
\begin{pmatrix}
R & B \\
2/5 & 3/5 \\
1/5 & 4/5
\end{pmatrix}
\]
\[
\begin{pmatrix}
2/5 & 3/5 \\
1/5 & 4/5
\end{pmatrix}^2 = \begin{pmatrix}
7/25 & 18/25 \\
6/25 & 19/25
\end{pmatrix}
\]
So, the probability is 18/25.

11. \[
\begin{pmatrix}
0.6 & 0.4 \\
0.37 & 0.63
\end{pmatrix}
\begin{pmatrix}
0.18 & 0.82
\end{pmatrix}
= \begin{pmatrix}
0.294 & 0.706
\end{pmatrix}
\]
So, 29.4% graduate as premed.

12. Suppose the rows of \(p^n\) converges to the matrix \(P\) where each row of \(P\) is \(\pi\). If we start at timestep 0 with distribution \(q\), then the distribution at timestep \(n\) is \(qp^n\). As \(n\) gets large, \(qp^n \rightarrow qP\). The first entry of \(qP\) is \(q_1\pi_1 + q_2\pi_1 + ... + q_r\pi_r = (q_1 + ... + q_r)\pi_1 = \pi_1\), and similarly for the other entries of \(qP\).

13. \(p = \begin{pmatrix}
1/4 & 3/4 \\
1/5 & 4/5
\end{pmatrix}\).

We can get from every state to every other state, so \(p\) is irreducible. Also, \(p(T, T) > 0\), so the state \(T\) is aperiodic.

Thus \(p\) has a unique standard distribution and the rows of \(p^n\) converge to it.

The standard distribution is the last row of \(\begin{pmatrix}
-3/4 & 1 \\
1/5 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
-20/19 & 20/19 \\
4/19 & 15/19
\end{pmatrix}\).

So, 4/19 of the vehicles are trucks.

14. \(p = \begin{pmatrix}
0 & 1/2 & 1/2 \\
1/4 & 1/2 & 1/4 \\
1/4 & 1/4 & 1/2
\end{pmatrix}\).

We can get from every state to every other state, so \(p\) is irreducible. Also, \(p(\text{rainy, rainy}) > 0\), so the state rainy is aperiodic.

Thus \(p\) has a unique standard distribution and the rows of \(p^n\) converge to it.

We have the following equations:

\(-\pi_1 + \pi_2/4 + \pi_3/4 = 0 \\
\pi_1/2 - \pi_2/2 + \pi_3/4 = 0 \\
\pi_1 + \pi_2 + \pi_3 = 1\)

When we solve, we have \(\pi_1 = 1/5, \pi_2 = 2/5, \pi_3 = 2/5\).

So 1/5 of the days are sunny, 2/5 are cloudy, and 2/5 are rainy.
The last state has period 2. The first state has period 3.

16. \( r = \begin{bmatrix} 0.2 & 0.6 \\ 0 & 0.55 \end{bmatrix} \)

\[
I - r = \begin{bmatrix} 0.8 & -0.6 \\ 0 & 0.45 \end{bmatrix}
\]

\[
(I - r)^{-1} = \begin{bmatrix} 5/4 & 5/3 \\ 0 & 20/9 \end{bmatrix}
\]

\[
n = \begin{bmatrix} 0 & 0.2 \\ 0.15 & 0.3 \end{bmatrix}
\]

\[
(I - r)^{-1}n = \begin{bmatrix} 5/4 & 5/3 \\ 0 & 20/9 \end{bmatrix} \begin{bmatrix} 0 & 0.2 \\ 0.15 & 0.3 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 1/3 & 2/3 \end{bmatrix}
\]

1/4 of trainees become supervisors. The expected time for a trainee to quit or become a supervisor is \( 2(5/4 + 5/3) = 35/6 \) months.