Final Review

Math 4653, Fall 2017

December 12, 2017

1. Given that $\text{cov}(X,Y) = 0$, which of the following do we know to be true? Circle all correct answers.
   
   (a) $E(X + Y) = EX + EY$.
   (b) $E(XY) = (EX)(EY)$.
   (c) $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$.
   (d) $\text{var}(XY) = (\text{var}(X))(\text{var}(Y))$.

2. Which of the following are distribution functions? Circle all correct answers.
   
   (a) $F(x) = \begin{cases} 0 & x < 1, \\ 1 & x \geq 1. \end{cases}$
   (b) $F(x) = \begin{cases} e^x & x \leq 0, \\ 1 & x > 0. \end{cases}$
   (c) $F(x) = \begin{cases} -1/x & x < -1, \\ x^2 & -1 \leq x \leq 1, \\ 1 - 1/x & x > 1. \end{cases}$
   (d) $F(x) = \begin{cases} 0 & x < -1, \\ 1/2 & -1 \leq x \leq 1, \\ 1 & x > 1. \end{cases}$
   (e) $F(x) = 1$.

3. A die is rolled repeatedly until the sum of the numbers obtained is larger than 200. What is the probability that you need more than 66 rolls to do this?

4. Let
   
   $F(x) = \begin{cases} e^{-1/x} & x > 0, \\ 0 & x \leq 0. \end{cases}$

   Is $F$ a distribution function? If so, find its density function.
5. Of the first 10000 votes cast in an election, 5180 were for candidate A. Find a 95% confidence interval for the fraction of votes candidate A will receive.

6. Suppose $X$ has density function

$$f(x) = \begin{cases} 6x(1-x) & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find $EX$ and $\text{var}(X)$.

7. Show that if $X = \text{exponential}(1)$ then $Y = X/\lambda$ is exponential($\lambda$).

8. Consider

$$f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

What value of $c$ makes $f$ a density function?

9. We have two coins. One is fair and the other produces heads with probability 3/4. One of the two coins is picked at random, and this coin is tossed $n$ times. After we observe a large number of tosses, can we tell which coin was chosen?

10. Suppose $X$ has density function

$$f(x) = \begin{cases} 4x^3 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution function and $P(1/3 < X < 2/3)$.

11. Suppose $X$ and $Y$ have joint density

$$f(x,y) = \begin{cases} \frac{5}{2}(x^2 + y) & 0 \leq x, \ 0 \leq y \leq 1 - x^2, \\ 0 & \text{otherwise.} \end{cases}$$

What are the joint distribution, marginal densities, and conditional densities? Are $X$ and $Y$ independent?

12. Let $X$ be a random variable with $EX = 0$ and $\text{var}(X) = 1$. What integer value of $k$ will assure us the $P(|X| \geq k) \leq 0.01$?

13. In a 162-game season, find the probability that a team with a 0.55 chance of winning each game will win at least 100 games.

14. Two people agree to meet for a drink after work but they are impatient and each will wait only 15 minutes for the other person to show up. Suppose they each arrive at independent random times uniformly distributed between 5 P.M. and 6 P.M. What is the probability they will meet?
15. Suppose

\[ f_X(x) = \begin{cases} 
1 & 0 < x < 1, \\
0 & \text{otherwise},
\end{cases} \]

\[ f_Y(x) = \begin{cases} 
\lambda e^{-\lambda x} & 0 < x, \\
0 & \text{otherwise},
\end{cases} \]

and \( X \) and \( Y \) are independent. If \( R^2 = X^2 + Y^2 \), find \( f_{R^2} \) (you do not need to evaluate the integrals).

16. A casino owner is concerned that his dice show 6 too often. He makes his employees roll a die 18000 times and they observe 3123 sixes. Is the die biased?

17. Suppose \( X \) and \( Y \) have joint density

\[ f(x, y) = \begin{cases} 
6xy^2 & 0 < x, y < 1, \\
0 & \text{otherwise}.
\end{cases} \]

What is \( P(X + Y < 1) \)?

18. Suppose we draw 5 cards out of a deck of 52. What is the expected number of suites in our hand?

19. For a class project, you are supposed to take a poll to forecast the outcome of an election. How many people do you have to ask so that with probability 0.95 your estimate will not differ from the true outcome by more than 5%?