18. We are given that $P(a < X < b) = 1$.

$r(x) = cx + d$, which is continuous and strictly increasing because $c > 0$.

$r$ maps $(a, b)$ to $(ca + d, cb + d)$, so $\alpha = ca + d$ and $\beta = cb + d$.

If $y = cx + d$, then $(1/c)(y - d) = x$. So, $s(y) = (1/c)(y - d)$ is the inverse of $r$.

Thus, $Y$ has density $(1/c)f((1/c)(y - d))$ on $ca + d < y < cb + d$.

20. We are given that $P(0 < X < 1) = 1$.

$r(x) = x^n$, which is continuous and strictly increasing.

$r$ maps $(0, 1)$ to $(0, 1)$, so $\alpha = 0$ and $\beta = 1$.

If $y = x^n$, then $y^{1/n} = x$. So, $s(y) = y^{1/n}$ is the inverse of $r$.

Thus, $Y$ has density $(1/n)y^{(1-n)/n}f(y^{1/n}) = (1/n)y^{(1-n)/n}$ on $0 < y < 1$.

28.

\[
\int_0^1 \int_0^{1-x} 6xy^2 \, dy \, dx = \int_0^1 2xy^3 \Big|_0^{1-x} \, dx \\
= \int_0^1 2x(1 - x)^3 \, dx \\
= \int_0^1 2x - 6x^2 + 6x^3 - 2x^4 \, dx \\
= x^2 - 2x^3 + (3/2)x^4 - (2/5)x^5 \Big|_0^1 \\
= 1 - 2 + (3/2) - (2/5) \\
= 1/10
\]
29. First I find my bounds of integration by drawing a picture.

\[
P(X - Y > z) = \int_{z}^{1} \int_{0}^{x-z} 2 \, dy \, dx
\]

\[
= \int_{z}^{1} 2y \big|_{0}^{x-z} \, dx
\]

\[
= \int_{z}^{1} 2x - 2z \, dx
\]

\[
= \int_{z}^{1} x^2 - 2xz \, dx
\]

\[
= x^2 - 2xz \big|_{z}^{1}
\]

\[
= 1 - 2z - (z^2 - 2z^2)
\]

\[
= 1 - 2z + z^2 \text{ for } 0 < z < 1
\]

32.

\[
\int_{0}^{x} \int_{0}^{y} e^{-(u+v)} \, dv \, du = \int_{0}^{x} \int_{0}^{y} e^{-u} e^{-v} \, dv \, du
\]

\[
= \int_{0}^{x} e^{-u} \int_{0}^{y} e^{-v} \, dv \, du
\]

\[
= \int_{0}^{x} e^{-u} (1 - e^{-y}) \, du
\]

\[
= (1 - e^{-y}) \int_{0}^{x} e^{-u} \, du
\]

\[
= (1 - e^{-y})(1 - e^{-x})
\]

\[
F(x, y) = \begin{cases} 
(1 - e^{-y})(1 - e^{-x}) & 0 < x, y, \\
0 & \text{otherwise.}
\end{cases}
\]

34.

\[
\frac{\partial^2}{\partial x \partial y} (x^3 y^2) = \frac{\partial}{\partial y} (3x^2 y^2) = 6x^2 y \text{ for } 0 \leq x, y \leq 1
\]
36. (a)

\[
f_X(x) = \int_0^\infty xe^{-x(1+y)} \, dy \\
= \int_0^\infty xe^{-x} e^{-xy} \, dy \\
= xe^{-x} \int_0^\infty e^{-xy} \, dy \\
= xe^{-x} \left. \frac{e^{-xy}}{-y} \right|_0^\infty \\
= -e^{-x} e^{-xy} \bigg|_0^\infty \\
= e^{-x}
\]

\[
f_Y(y) = \int_0^\infty xe^{-x(1+y)} \, dx \\
= xe^{-x(1+y)} \bigg|_{-1}^{\infty} - \int_0^\infty e^{-x(1+y)} \, dx \\
= 0 + \frac{1}{1+y} \int_0^\infty e^{-x(1+y)} \, dx \\
= \frac{1}{1+y} e^{-x(1+y)} \bigg|_0^\infty \\
= \frac{1}{1+y} \left( 0 - \frac{1}{-(1+y)} \right) \\
= \frac{1}{(1+y)^2}
\]

\[
f_X(0) = 1, f_Y(0) = 1, f(0,0) = 0. \text{ So they are not independent.}
\]

(b) They are not independent, as \( f > 0 \) on a triangle, not a rectangle.

Notice that even though it looks like \( f \) satisfies the conditions of 5.5, it doesn’t, because if \( g(x) = 6x \) on \( 0 < x < a \) and \( h(y) = y^2 \) on \( 0 < y < b \), then \( g(x)h(x) = 6xy^2 \) on \( 0 < x < a \) and \( 0 < y < b \), which is not the region on which \( f(x, y) > 0 \).

(c) They are independent by Theorem 5.5, letting \( g(x) = x \) on \( 0 < x < 1 \) and \( h(y) = 2y + 1 \) on \( 0 < y < 1 \).
37. (a)

\[ f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \]
\[ = \frac{1}{\pi} y\sqrt{1-x^2} \]
\[ = \frac{1}{\pi} \left( \sqrt{1-x^2} - (-\sqrt{1-x^2}) \right) \]
\[ = \frac{2\sqrt{1-x^2}}{\pi} \quad \text{on} \quad -1 < x < 1 \]

(b) \( f_Y(y \mid X = x) = \frac{1/\pi}{2\sqrt{1-x^2}/\pi} = \frac{1}{2\sqrt{1-x^2}} \) when \( x^2 + y^2 \leq 1 \).

39. (a)

\[ f_X(x) = \int_0^{1-x} 6y \, dy \]
\[ = 3y^2 \bigg|_0^{1-x} \]
\[ = 3(1-x)^2 \quad \text{for} \quad 0 < x < 1 \]

\[ f_Y(y) = \int_0^{1-y} 6x \, dx \]
\[ = 6yx \bigg|_0^{1-y} \]
\[ = 6y(1-y) \quad \text{for} \quad 0 < x < 1 \]

(b) \( f_Y(y \mid X = x) = \frac{6y}{3(1-x)^2} = \frac{2y}{(1-x)^2} \) when \( 0 < x, y \) and \( x + y < 1 \).

2. Let \( X_i \) be 1 if the \( i \)th card is a ace and 0 otherwise. We are looking for \( E(X_1 + \ldots + X_{13}) \).

\( EX_i = 1/13 \) for each \( i \), so the expected number of aces is \( 13(1/13) = 1 \).

3. \( X = \text{Binomial}(3, 10/25). \ Y = \text{Binomial}(3, 15/25) \).

\( E(X - Y) = EX - EY = 30/25 - 45/25 = -15/25 = -3/5 \).
11. 
\[ E(3X + 4Y - 5) = E(3X) + E(4Y - 5) \]
\[ = 3E(X) + 4E(Y) - 5 \]
\[ = 3 + 8 - 5 \]
\[ = 6 \]
\[ \text{var}(3X + 4Y - 5) = \text{var}(3X) + \text{var}(4Y - 5) \text{ by independence} \]
\[ = \text{var}(3X) + \text{var}(4Y) \]
\[ = 3^2 \text{var}(X) + 4^2 \text{var}(Y) \]
\[ = 3^2(3) + 4^2 \]
\[ = 43 \]

14. Let \( X_1 = N_1, X_2 = N_2 - N_1, X_3 = N_3 - N_2, \) and so on.

Then \( N_k = X_1 + \ldots + X_k. \)

Notice that each \( X_i \) is the amount of time it takes to get 1 success.

\[ EX_i = 1/p \text{ (see Example 1.23)} \]

So, \( EN_k = kEX_i = k/p. \)

Because the trials are independent, the \( X_i \)'s are independent.

\[ \text{var}(X_i) = \frac{1-p}{p^2} \text{ (see Example 1.28)} \]

So, \( \text{var}(N_k) = k \text{var}(X_i) = \frac{k(1-p)}{p^2}. \)

16. (a) \( EX = 0. \) \( EY = 1(2/5) + 4(2/5) = 2. \)

\[ \text{cov}(X,Y) = E((X)(Y - 2)) = E(X^3 - 2X) = E(X^3) - 2E(X) = 0. \]

(b) No. \( P(X = 1, Y = 0) = 0, \) but \( P(X = 1)P(Y = 0) = 1/25. \)

Additional Problem 1. Let \( X_1 \) be the number on the first die, \( X_2 \) be the number on the second die, and \( X_3 \) be the number on the third die.

<table>
<thead>
<tr>
<th>( X_2 \backslash X_1 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
</tr>
<tr>
<td>2</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
</tr>
<tr>
<td>3</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
</tr>
<tr>
<td>4</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
</tr>
<tr>
<td>5</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
</tr>
<tr>
<td>6</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
<td>1/36</td>
</tr>
</tbody>
</table>

The lower left to upper right diagonals of this table have \( X_1 + X_2 \) constant.
The lower left to upper right diagonals of this table have $X_1 + X_2 + X_3$ constant.

We wanted $P(X_1 + X_2 + x_3 > 9)$, so we add the appropriate values to get $\frac{5}{8}$.

Additional Problem 2. (a) First we compute the joint distribution:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$X_1 + X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X_1 + X_2 = k)$</td>
<td>$\frac{1}{6^3}$ $\frac{3}{6^3}$ $\frac{6}{6^3}$ $\frac{10}{6^3}$ $\frac{15}{6^3}$ $\frac{21}{6^3}$ $\frac{25}{6^3}$ $\frac{27}{6^3}$</td>
</tr>
</tbody>
</table>

The lower left to upper right diagonals of this table have $X_1 + X_2$ constant.

(b) The upper left to lower right diagonals of the joint distribution table have $Y - X$ constant.

Additional Problem 3. (a) If $f_X(x)f_Y(z - x) > 0$, then $3 < x < 5$ and $z - 5 < x < z - 3$.

\[
\int_3^{z-3} \frac{1}{2} (x - 3) \frac{1}{2} (z - x - 3) \, dx = \frac{1}{4} \int_3^{z-3} -x^2 + zx - 3z + 9 \, dx
\]
\[
= \frac{1}{4} \left[ \frac{-x^3}{3} + \frac{zx^2}{2} - 3zx + 9x \right]_3^{z-3}
\]
\[
= \frac{1}{24} (z - 6)^3
\]
\[ \int_{z-5}^{z} \frac{1}{2} (x - 3) \frac{1}{2} (z - x - 3) \, dx = \frac{1}{4} \left( \frac{-x^3}{3} + \frac{zx^2}{2} - 3zx + 9x \right) \bigg|_{z-5}^{z} \]
\[ = \frac{1}{24} (-z^3 + 18z^2 - 84z + 40) \]

\[ f_{X+Y}(z) = \begin{cases} 
0 & z \leq 6, \\
\frac{1}{24} (z - 6)^3 & 6 \leq z \leq 8 \\
\frac{1}{24} (-z^3 + 18z^2 - 84z + 40) & 8 \leq z \leq 10, \\
0 & 10 \leq z.
\end{cases} \]

(b) If \( f_X(x)f_Y(z - x) > 0 \), then \( 0 < x < 2 \) and \( z - 2 < x < z \).

\[ \int_{0}^{z} \frac{1}{2} \left( \frac{1}{2} (z - x) \right) \, dx = \frac{1}{4} \int_{0}^{z} z - x \, dx \]
\[ = \frac{1}{4} \left( zx - \frac{x^2}{2} \right) \bigg|_{0}^{z} \]
\[ = \frac{1}{4} \left( z^2 - \frac{z^2}{2} \right) \]
\[ = \frac{z^2}{8} \]

\[ \int_{z-2}^{2} \frac{1}{2} \left( \frac{1}{2} (z - x) \right) \, dx = \frac{1}{4} \left( zx - \frac{x^2}{2} \right) \bigg|_{z-2}^{2} \]
\[ = \frac{1}{4} \left( 2z - 2 \right) - \left( z(z - 2) - \frac{(z - 2)^2}{2} \right) \]
\[ = \frac{1}{8} (-z^2 + 4z) \]

\[ f_{X+Y}(z) = \begin{cases} 
0 & z \leq 0, \\
\frac{z^2}{8} & 0 \leq z \leq 2 \\
\frac{1}{8} (-z^2 + 4z) & 2 \leq z \leq 4, \\
0 & 4 \leq z.
\end{cases} \]

(c) \( E = (2a, 2b) \) where \( \{ z \mid f_X(z) > 0 \} = \{ z \mid f_Y(z) > 0 \} = (a, b) \).

Additional Problem 4. (a) Notice that \( f(x, y) = g(x)h(y) \) where \( g(x) = xe^{-x} \) on \( 0 < x < \infty \) and \( h(y) = \frac{ey}{e-1} \) on \( 0 < y < 1 \).

This means that by Theorem 5.5, \( X \) and \( Y \) are independent and \( f_X(x) = cx e^{-x} \) on \( 0 < x < \infty \) and \( f_Y(y) = \frac{ey}{c(e-1)} \) on \( 0 < y < 1 \) for some constant \( c \).
We know that we must have $1 = \int_0^\infty cxe^{-x} \, dx$, so:

$$1 = \int_0^\infty cxe^{-x} \, dx$$

$$1 = cxe^{-x}\bigg|_0^\infty - \int_0^\infty -ce^{-x} \, dx$$

$$1 = -ce^{-x}\bigg|_0^\infty$$

$$1 = c$$

Thus, $f_X(x) = xe^{-x}$ on $0 < x < \infty$ and $f_Y(y) = \frac{e^y}{e-1}$ on $0 < y < 1$.

$$EX = \int_0^\infty x^2e^{-x} \, dx$$

$$= x^2e^{-x}\bigg|_0^\infty - \int_0^\infty 2xe^{-x} \, dx$$

$$= 2 \int_0^\infty xe^{-x} \, dx$$

$$= 2$$

$$EY = \int_0^1 ye^y \, dy$$

$$= \frac{1}{e-1} \int_0^1 ye^y \, dy$$

$$= \frac{1}{e-1} \left( ye^y|_0^1 - \int_0^1 e^y \, dy \right)$$

$$= \frac{1}{e-1} \left( e - e^1|_0^1 \right)$$

$$= \frac{1}{e-1} (e - e + 1)$$

$$= \frac{1}{e-1}$$

$$E(X + Y) = EX + EY$$

$$= 2 + \frac{1}{e-1}$$

$$= \frac{2e-1}{e-1}$$

(b) We determined above that $X$ and $Y$ are independent, so $E(XY) = (EX)(EY) = \frac{2}{e-1}$.
(c) Again, since $X$ and $Y$ are independent, $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$.

\[
EX^2 = \int_0^\infty x^3 e^{-x} \, dx
\]
\[
= x^3 e^{-x} \bigg|_0^\infty - \int_0^\infty -3x^2 e^{-x} \, dx
\]
\[
= 3 \int_0^\infty x^2 e^{-x} \, dx
\]
\[
= 6
\]

\[
EY^2 = \int_0^1 \frac{y^2 e^y}{e-1} \, dy
\]
\[
= \frac{1}{e-1} \int_0^1 y^2 e^y \, dy
\]
\[
= \frac{1}{e-1} \left( y^2 e^y \bigg|_0^1 - 2 \int_0^1 y e^y \, dy \right)
\]
\[
= \frac{1}{e-1} (e - 2)
\]

\[
\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)
\]
\[
= EX^2 - (EX)^2 + EY^2 - (EY)^2
\]
\[
= 6 - 2^2 + \frac{e - 2}{e - 1} - \frac{1}{(e - 1)^2}
\]
\[
= 2 + \frac{(e - 2)(e - 1)}{(e - 1)^2} - \frac{1}{(e - 1)^2}
\]
\[
= \frac{2e^2 - 4e + 2}{(e - 1)^2} + \frac{e^2 - 3e + 2}{(e - 1)^2} - \frac{1}{(e - 1)^2}
\]
\[
= \frac{3e^2 - 7e + 3}{(e - 1)^2}
\]