1. Find the number of zeros of the polynomial $z^4 + 3z^2 + z + 1$ inside the unit disk.

Let $f(z) = z^4 + 3z^2 + z + 1$ and $g(z) = 3z^2 + 1$.

The unit circle is a simple, closed, smooth path.

For any point on the unit circle, $|f(z) - g(z)| = |z^4 + z| \leq |z|^4 + |z| = 2 \leq |3z^2 + 1| = |g(z)|.$

Note that $|g(z)| = 2$ on the unit circle exactly when $z^2 = -1$, or $z = \pm i$.

$|f(i) - g(i)| = |i^4 + i| = |1 + i| = \sqrt{2} < 2.$

$|f(-i) - g(-i)| = |(-i)^4 - i| = |1 - i| = \sqrt{2} < 2.$

So, in all cases on the unit circle, $|f(z) - g(z)| < |g(z)| \leq |g(z)| + |f(z)|$.

So, by Rouché’s Theorem, $f$ and $g$ have the same number of zeros inside the unit circle.

$$g(z) = 0$$

$$3z^2 + 1 = 0$$

$$z^2 = -1/3$$

$$z = \pm i/\sqrt{3}$$

So, $f(z)$ has two zeros in the unit disk.

2. Let $n \geq 2$ be an integer. Evaluate the integral $\int_0^\infty \frac{1}{1+x^n} \, dx$. Carefully justify all your steps.

Consider the simple, closed, piecewise smooth path that travels from 0 to $R$ along the real axis, moves along the circle of radius $R$ in the counter-clockwise direction to $Re^{2\pi i/n}$, and then travels back to the origin in a straight line.

Call this path $\gamma_R$.

The interior of $\gamma_R$ contains one singularity of $\frac{1}{1+x^n}, e^{\pi i/n}$.
\[
\int_{\gamma_R} \frac{1}{1 + x^n} \, dx = 2\pi i \text{ Res}_{x = e^{\pi i/n}} \frac{1}{1 + x^n} \text{ by the Residue Theorem}
\]
\[
= 2\pi i \frac{1}{n(e^{\pi i/n})^{n-1}} \text{ as } \text{Res}_{z=z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)} \text{ if } h(z_0) = 0 \text{ and } g(z_0), h'(z_0) \neq 0
\]
\[
= \frac{2\pi i}{ne^{(n-1)\pi i/n}}
\]

\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{1}{1 + x^n} \, dx \right| \leq \lim_{R \to \infty} \int_{C_R} \left| \frac{1}{1 + x^n} \right| \, dx
\]
\[
\leq \lim_{R \to \infty} \int_{C_R} \frac{1}{2\pi R} \, dx
\]
\[
= \lim_{R \to \infty} \frac{2\pi R}{R^n - 1}
\]
\[
= 0
\]
\[ \lim_{R \to \infty} \int_{\gamma_R} \frac{1}{1 + x^n} \, dx = \lim_{R \to \infty} \left( \int_0^R \frac{1}{1 + x^n} \, dx + \int_{C_R} \frac{1}{1 + x^n} \, dx + \int_{R e^{2 \pi i / n}}^0 \frac{1}{1 + x^n} \, dx \right) \]
\[ = \lim_{R \to \infty} \left( \int_0^R \frac{1}{1 + x^n} \, dx + \int_0^0 \frac{1}{1 + x^n} \, dx \right) + \lim_{R \to \infty} \int_{C_R} \frac{1}{1 + x^n} \, dx \]
\[ = \lim_{R \to \infty} \left( \int_0^R \frac{1}{1 + x^n} \, dx + \int_0^R \frac{1}{1 + (e^{2 \pi i / n} x)^n} e^{2 \pi i / n} \, dx \right) + 0 \]

by a change of variables
\[ = \lim_{R \to \infty} \left( \int_0^R \frac{1}{1 + x^n} \, dx - e^{2 \pi i / n} \int_0^R \frac{1}{1 + e^{2 n \pi i / n} x^n} \, dx \right) \]
\[ = \lim_{R \to \infty} \left( \int_0^R \frac{1}{1 + x^n} \, dx \right) + \lim_{R \to \infty} \left( \int_0^R \frac{1}{1 + x^n} \, dx \right) \]
\[ = \lim_{R \to \infty} \left( \int_0^R \frac{1}{1 + x^n} \, dx \right) \]

3. Suppose \( f \) is analytic on \( \{ z \mid 0 < |z| < 1 \} \) and \( |f(z)| \leq \log \left( \frac{1}{|z|} \right) \). Show that \( f \) is identically 0.

\[ \lim_{z \to 0} |zf(z)| \leq \lim_{z \to 0} |z| \log \left( \frac{1}{|z|} \right) = 0. \]

This means \( f \) has a removable singularity at 0.

So, \( \exists g \) holomorphic in \( \{ z \mid |z| < 1 \} \) where \( g|_{\{ z \mid 0 < |z| < 1 \}} = f \).

Choose any \( w \) with \( |w| < 1 \).
\[ |g(w)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{z-w} \, dz \right| \text{ with } \gamma \text{ any circle centered at 0 containing } w \]

\[ \leq \frac{1}{2\pi} \int_{\gamma} \left| \frac{g(z)}{z-w} \right| \, dz \]

\[ \leq \frac{1}{2\pi} \int_{\gamma} \frac{\log \left( \frac{1}{|z-w|} \right)}{z-w} \, dz \]

\[ \leq \frac{1}{2\pi} \int_{\gamma} \log \left( \frac{1}{R} \right) \, dz \text{ if } R \text{ is the radius of the circle } \gamma \]

\[ = \frac{1}{2\pi} 2\pi R \log \left( \frac{1}{R} \right) \]

\[ = \frac{R}{R-|w|} \log \left( \frac{1}{R} \right) \]

\[ \to 0 \text{ as } R \to 1 \]

So, \( g \) is identically 0, which means \( f \) is identically 0.

4. Show that the equation \( \sin(z) = z \) has infinitely many solutions in the complex plane.

We know that \( \sin(z) - z \) has an essential singularity at \( \infty \).

By Picard’s Great Theorem, there is at most one value, call it \( w \), that \( \sin(z) - z \) does not take infinitely many times.

\( \sin(z) - z \) must take the value \( w + 2\pi \) infinitely many times.

But if \( \sin(z) - z = w + 2\pi \), then \( \sin(z + 2\pi) - (z + 2\pi) = \sin(z) - z - 2\pi = w \).

So, \( \sin(z) - z \) takes the value \( w \) infinitely many times.

Sine \( \sin(z) - z \) takes every value infinitely many times, \( \sin z - z = 0 \) infinitely many times.

Thus, \( \sin(z) - z \) has infinitely many solutions.

5. (a) State Schwarz’s Lemma.

Let \( D \) be the open unit disk. If \( f : D \to D \) is an analytic function fixing 0, then

(i) \( \forall z \in D, |f(z)| \leq |z| \) and \( |f'(0)| \leq 1 \).

(ii) Further, if \( |f'(0)| = 1 \) or \( |f(z)| = |z| \) for any \( z \in D \setminus \{0\} \), then \( f(z) = \alpha z \) for some \( \alpha \) with \( |\alpha| = 1 \).

(b) Let \( f : D \to D \) be a holomorphic map of the unit disk to itself. Prove that for all \( z \in D, \frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \).
Note that the Möbius transformation \( \frac{z_0 - z}{1 - z_0 z} \) maps \( D \) to \( D \) and interchanges \( z_0 \) and 0.

Fix \( z_0 \in D \).

Define \( g(z) = \frac{z_0 - z}{1 - z_0 z}, h(z) = \frac{f(z_0) - z}{1 - f(z_0)z} \).

Then \( h \circ f \circ g^{-1} \) is holomorphic from \( D \) to itself and fixes 0.

\[
\left| \frac{f(z_0) - f(g^{-1}(z))}{1 - f(z_0)f(g^{-1}(z))} \right| \leq |z| \quad \text{by Schwarz's Lemma}
\]

\[
\left| \frac{f(z_0) - f(w)}{1 - f(z_0)f(w)} \right| \leq \frac{|z_0 - w|}{1 - z_0 w} \quad \text{letting } w = g^{-1}(z)
\]

\[
\left| \frac{f(z_0) - f(w)}{(z_0 - w)(1 - f(z_0)f(w))} \right| \leq \frac{1}{1 - z_0 w}
\]

\[
\left| \frac{f'(z_0)}{1 - |f(z_0)|^2} \right| \leq \frac{1}{1 - |z|^2} \quad \text{letting } w \rightarrow z_0
\]

Since \( z_0 \) was arbitrary, \( \left| \frac{f'(z)}{1 - |f(z)|^2} \right| \leq \frac{1}{1 - |z|^2} \quad \forall z \in D \) (we can drop some of the absolute values because \( |z|, |f(z)| < 1 \)).

6. Determine all continuous functions on \( \{ z \mid 0 < |z| \leq 1 \} \) which are harmonic on \( \{ z \mid 0 < |z| < 1 \} \) and which are identically 0 on \( \{ z \mid |z| = 1 \} \).

Complex-valued harmonic functions on an annulus are of the form \( a_0 + b_0 \log|z| + \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k z^k + b_k z^{-k} \).

If \( |z| = 1, z = e^{i\theta} \).

\[
0 = a_0 + b_0 \ln 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k e^{ik\theta} + b_k e^{-ik\theta} = a_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}} (a_k + b_k) e^{ik\theta}.
\]

Then \( \sum_{k \in \mathbb{Z} \setminus \{0\}} (a_k + b_k) e^{ik\theta} \) is constant, so \( \sum_{k \in \mathbb{Z} \setminus \{0\}} ik(a_k + b_k) e^{ik\theta} = 0 \).

But the \( e^{ik\theta} \) are orthogonal, so the only way for this to be true is for \( k(a_k + b_k) = 0 \ \forall k \).

This means \( a_0 = 0 \) as well.

\( f(z) = b \log|z| \).

7. (a) Prove that the series \( \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \) converges to a meromorphic function on \( \mathbb{C} \).

We will first prove \( \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \) is holomorphic away from \( \mathbb{Z} \).
Let $K$ be compact in $\mathbb{C} \setminus \mathbb{Z}$.

$\exists N_1, N_2$ such that $N_1 \leq \text{Re}(z) \leq N_2 \ \forall z \in K$.

Let $m = \min\{|z - n| \mid n \in \mathbb{Z}\}$.

If $n < N_1$, $\left| \frac{1}{(z - n)^2} \right| \leq \frac{1}{|N_1 - n|^2}$.

If $n > N_2$, $\left| \frac{1}{(z - n)^2} \right| \leq \frac{1}{|N_2 - n|^2}$.

If $N_1 \leq n \leq N_2$, $\left| \frac{1}{(z - n)^2} \right| \leq \frac{1}{m^2}$.

$\sum_{n=-\infty}^{N_1-1} \frac{1}{|N_1 - n|^2} + \sum_{n=N_1}^{N_2} \frac{1}{m^2} + \sum_{n=N_2+1}^{\infty} \frac{1}{|N_2 - n|^2} < \infty$.

So, by the Weierstrass M-test, $\sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}$ converges uniformly on $K$.

Then by Weierstrass’ Theorem, $\sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}$ is holomorphic in $\mathbb{C} \setminus \mathbb{Z}$.

Every element of $\mathbb{Z}$ is a pole, because for a given $n_0 \in \mathbb{Z}$, $\lim_{z \to n_0} \sum_{n=-\infty}^{\infty} \frac{(z - n_0)^2}{(z - n)^2} = 1$.

So, the function is meromorphic.

(b) Prove that there is an entire function $h(z)$ so that $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2} + h(z)$.

Let $g(z) = \frac{\pi^2}{\sin^2(\pi z)} - \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}$.

$g$ has singularities at the integers and is holomorphic everywhere else.

We want to show that $g$’s singularities are removable.

Since $g$ is periodic with period 1, we only need to look at one singularity of $g$, we will choose the singularity at 0.

Since $\frac{\pi^2}{\sin^2(\pi z)}$ is even, the singular part cannot be $\frac{1}{z}$.

So we check $\frac{1}{z^2}$:

$$
\lim_{z \to 0} \frac{z^2 \pi^2}{\sin^2(\pi z)} = \lim_{z \to 0} \frac{2z \pi^2}{2 \pi \sin(\pi z) \cos(\pi z)} = \lim_{z \to 0} \frac{z \pi}{\sin(\pi z) \cos(\pi z)} = \lim_{z \to 0} \frac{\pi}{\pi \cos(\pi z) \cos(\pi z) - \pi \sin(\pi z) \sin(\pi z)} = 1
$$

The singular part of $g$ at $n \in \mathbb{Z}$ is $\frac{1}{(z - n)^2}$.

The same is true of $\sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}$. 


So, $g$ has only removable singularities.

$\exists h$ entire where $h|_{\mathbb{C}\setminus Z} = g$.

So, $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + h(z)$.

8. Show that the total number of poles of an elliptic function $f$ in its fundamental parallelogram is $\geq 2$.

There cannot be 0 poles in the fundamental parallelogram, because then $f$ would be bounded and, by Liouville’s Theorem, constant.

Assume for contradiction that there is only one pole.

Then $\int_{\partial P} f \neq 0$ by the Residue Theorem.

\[
\int_{\partial P} f = \int_a^{a+\omega_1} f + \int_a^{a+\omega_1+\omega_2} f + \int_a^{a+\omega_2} f + \int_a^{a+\omega_2} f
\]

\[
= \int_a^{a+\omega_1} f + \int_a^{a+\omega_1+\omega_2} f - \int_a^{a+\omega_1+\omega_2} f - \int_a^{a+\omega_2} f
\]

\[
= 0 \text{ by double periodicity}
\]

This is a contradiction, so an elliptic function must have at least 2 poles in its fundamental parallelogram.