1. Prove that there is no one-to-one conformal map of the punctured disc \( \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \) onto the annulus \( \{ z \in \mathbb{C} \mid 1 < |z| < 2 \} \).

Suppose there exists such a map, \( f \).

Then \( f \) is bounded in the punctured disk, so \( \lim_{z \to 0} zf(z) = 0 \).

So, 0 is a removable singularity of \( f \).

Consider then \( g \), which is holomorphic on the unit disc and agrees with \( f \) on the punctured disc.

First suppose \( g(0) = w \) in the annulus \( \{ z \in \mathbb{C} \mid 1 < |z| < 2 \} \).

\( \exists z \neq 0 \) in the punctured disc where \( g(z) = w \), because \( f \) is onto.

Let \( A_1 \) be a neighborhood of 0, \( A_2 \) be a neighborhood of \( z \) where \( A_1 \cap A_2 = \emptyset \).

Then \( g(A_1), g(A_2) \) are open by the Open Mapping Theorem.

\( w \in g(A_1) \cap g(A_2) \), so \( g(A_1) \cap g(A_2) \) is a nonempty open set.

\( \exists w' \neq w \in g(A_1) \cap g(A_2) \).

This means \( \exists a, b \) in the punctured disc where \( f(a) = f(b) = w' \).

This is a contradiction, because \( f \) is one-to-one.

Then \( g(0) \) must be on the border of the annulus \( \{ z \in \mathbb{C} \mid 1 < |z| < 2 \} \).

This is impossible, because, by the Open Mapping Theorem, the image of the unit disc under \( g \) must be open or only a point.

So, no such \( f \) exists.

2. (a) State Schwarz’s Lemma.

Let \( D \) be the open unit disk. If \( f : D \to D \) is an analytic function fixing 0, then

(i) \( \forall z \in D, |f(z)| \leq |z| \) and \( |f'(0)| \leq 1 \).

(ii) Further, if \( |f'(0)| = 1 \) or \( |f(z)| = |z| \) for any \( z \in D \setminus \{0\} \), then \( f(z) = \alpha z \) for some \( \alpha \) with \( |\alpha| = 1 \).
(b) Let \( f : D \rightarrow D \) be a holomorphic map of the unit disc into itself. Prove that for all \( z \in D \),

\[
\left| f'(z) \right| \leq \frac{1}{1 - |f(z)|^2}.
\]

Note that the Möbius transformation \( \frac{z_0 - z}{1 - z_0z} \) maps \( D \) to \( D \) and interchanges \( z_0 \) and 0.

Fix \( z_0 \in D \).

Define \( g(z) = \frac{z_0 - z}{1 - z_0z}, h(z) = \frac{f(z_0) - z}{1 - f(z_0)z} \).

Then \( h \circ f \circ g^{-1} \) is holomorphic from \( D \) to itself and fixes 0.

\[
\left| \frac{f(z_0) - f(g^{-1}(z))}{1 - f(z_0)f(g^{-1}(z))} \right| \leq |z| \quad \text{by Schwarz's Lemma}
\]

\[
\left| \frac{f(z_0) - f(w)}{1 - f(z_0)f(w)} \right| \leq \frac{|z_0 - w|}{1 - \overline{z_0w}} \quad \text{letting } w = g^{-1}(z)
\]

\[
\left| \frac{f(z_0) - f(w)}{(z_0 - w)(1 - f(z_0)f(w))} \right| \leq \frac{1}{1 - \overline{z_0w}}
\]

\[
\left| \frac{f'(z_0)}{1 - |f(z_0)|^2} \right| \leq \frac{1}{1 - |z_0|^2} \quad \text{letting } w \rightarrow z_0
\]

Since \( z_0 \) was arbitrary, \( \left| f'(z) \right| \leq \frac{1}{1 - |z|^2} \quad \forall z \in D \) (note we can drop some of the absolute values because \( |z|, |f(z)| < 1 \)).

3. Prove that for any \( a \in \mathbb{C} \) and \( n \geq 2 \), the polynomial \( az^n + z + 1 \) has at least one root in the disc \( |z| \leq 2 \).

Rewrite \( az^n + z + 1 \) as \( a(z - \omega_1)...(z - \omega_n) \).

Then \( |a||\omega_1|...|\omega_n| = 1 \).

\[
|a| = \frac{1}{|\omega_1|...|\omega_n|}.
\]

If \( |a| \geq \frac{1}{2^{|n|}} \), then at least one of the \( \omega_k \)'s must be in the disc \( |z| \leq 2 \).

If \( |a| < \frac{1}{2^{|n|}} \), we will apply Rouché's Theorem.

Let \( f(z) = az^n + z + 1, g(z) = z + 1 \).

On the boundary of the disc \( |z| < 2 \), we get the following inequality:
\[ |f(z) - g(z)| = |az^n + z + 1 - z - 1| \\
= |az^n| \\
= |a|2^n \\
< 1 \\
\leq |z + 1| \\
= |g(z)| \\
\leq |f(z)| + |g(z)|

So, \( f \) and \( g \) have the same number of zeros in the disc \( |z| < 2 \).

\( g \) has 1 zero in this disc, so \( f \) does as well.
Thus, in either case, \( az^n + z + 1 \) has at least one zero in the disc \( |z| < 2 \).

4. Evaluate the integral \( \int_{0}^{\infty} \frac{x^2}{1 + x^6} dx \). Carefully justify all your steps.

Let \( \gamma_R \) be the curve that goes from \(-R\) to \(R\) along the real axis and then back to \(-R\) along the upper semicircle.

\[
\int_{\gamma_R} \frac{x^2}{1 + x^6} dx = 2\pi i \left( \text{Res}_{x=e^{\pi i/6}} \frac{x^2}{1 + x^6} + \text{Res}_{x=i} \frac{x^2}{1 + x^6} + \text{Res}_{x=e^{5\pi i/6}} \frac{x^2}{1 + x^6} \right)
\]
\[
= 2\pi i \left( \frac{e^{2\pi i/6}}{6e^{5\pi i/6}} + \frac{e^{6\pi i/6}}{6e^{15\pi i/6}} + \frac{e^{10\pi i/6}}{6e^{25\pi i/6}} \right)
\]
\[
= \pi i \left( \frac{e^{2\pi i/6}e^{15\pi i/6} + e^{6\pi i/6}e^{5\pi i/6} + e^{10\pi i/6}e^{5\pi i/6}e^{15\pi i/6}}{e^{5\pi i/6}e^{15\pi i/6}e^{25\pi i/6}} \right)
\]
\[
= \frac{\pi i}{3} \left( -1 + 1 - 1 \right)
\]
\[
= \frac{\pi}{3}
\]
\begin{align*}
\left| \int_{C_R} \frac{x^2}{1 + x^6} \, dx \right| &\leq \int_{C_R} \left| \frac{x^2}{1 + x^6} \right| \, dx \\
&\leq \int_{C_R} \frac{R^2}{|R^6 - 1|} \, dx \\
&= \frac{2\pi R^3}{R^6 - 1} \, dx \\
&\to 0 \text{ as } R \to \infty
\end{align*}

\begin{align*}
\frac{1}{2} \lim_{R \to \infty} \int_{\gamma_R} \frac{x^2}{1 + x^6} \, dx &= \frac{1}{2} \lim_{R \to \infty} \left( \int_{-R}^{R} \frac{x^2}{1 + x^6} \, dx + \int_{C_R} \frac{x^2}{1 + x^6} \, dx \right) \\
\frac{1}{2} \lim_{R \to \infty} \left( \frac{\pi}{3} \right) &= \frac{1}{2} \left( \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2}{1 + x^6} \, dx + \lim_{R \to \infty} \int_{C_R} \frac{x^2}{1 + x^6} \, dx \right) \\
\frac{1}{2} \left( \frac{\pi}{3} \right) &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{1 + x^6} \, dx \\
\frac{\pi}{6} &= \int_{0}^{\infty} \frac{x^2}{1 + x^6} \, dx \text{ as } \frac{x^2}{1 + x^6} \text{ is even}
\end{align*}

5. (a) State Cauchy-Goursat’s Theorem.

If \( f \) is analytic on an open set \( D \) and \( C \) is a closed, simple curve inside \( D \), then

\[ \int_{C} f = 0. \]

(b) Use Cauchy-Goursat’s Theorem to prove that if the function \( f \) is continuous on \( \mathbb{C} \) and analytic on every point not on the real axis, then \( f \) is analytic everywhere.

Let \( C \) be any closed, simple curve in \( \mathbb{C} \).

If \( C \) is entirely in the upper or lower half-plane, then, by Cauchy-Goursat’s Theorem,

\[ \int_{C} f = 0. \]

If \( C \) crosses the real axis, we evaluate the integral by cutting \( C \) along the real axis, and adding segments \( \varepsilon \) away from the real axis in both the upper and lower half-planes.

The integrals of each of these pieces are 0 because each piece is only in one half-plane. Adding up the integrals and letting \( \varepsilon \to 0 \), the parts along the real line cancel out, and we find that \( \int_{C} f = 0. \)

Since this is true for any closed, simple \( C \), by Morera’s Theorem, \( f \) is analytic on \( \mathbb{C} \).
6. Prove that if the composition of \( f \circ g \) of two entire functions \( f \) and \( g \) is a polynomial, then both \( f \) and \( g \) are polynomials.

This statement is only true if constants are not polynomials (otherwise, for example, \( f(z) = e^z \), \( g(z) = 1 \) would be a contradiction).

Let \( m \) be the order of \( f \circ g \).

Assume at least one of \( f, g \) is not a polynomial.

Since they are entire, \( f = \sum_{k=0}^{\infty} a_k z^k \), \( g = \sum_{k=0}^{\infty} b_k z^k \).

\[
f \circ g = f \left( \sum_{k=0}^{\infty} b_k z^k \right) = \sum_{k=0}^{\infty} a_k \left( \sum_{k=0}^{\infty} b_k z^k \right)^k
\]

Either \( f \) or \( g \) has a nonzero \( z^n \) term, where \( n > m \), and the other has a nonzero \( z^{\ell} \) term where \( \ell > 0 \).

Then \( f \circ g \) has a nonzero \( z^{n\ell} \) term, but \( n\ell > m \) and this is a contradiction.

7. Suppose \( f(z) \) is analytic on the unit disc \( D(0, 1) \) and continuous on the closed unit disc \( \overline{D}(0, 1) \). Assume that \( f(z) = 0 \) on an arc of the circle \( z = 1 \). Show that \( f(z) \equiv 0 \).

Let \( h(z) = \frac{z-i}{z+i} \), which bijectively and analytically maps the upper half plane, \( \mathbb{H} \), to the unit disc \( D \).

Let \( g = f \circ h \).

\( g \) is an analytic map from \( \mathbb{H} \) to \( D \) that is continuous on \( \mathbb{H} \cup \mathbb{R} \) and sends some interval of \( \mathbb{R} \) to 0.

Let \( U \) be an open set of \( \mathbb{H} \) where \( U \) intersects \( \mathbb{R} \) on some interval \( I \).

Let \( U' \) be the reflection of \( U \) across the real axis.

Then \( \exists G \) from \( U \cup I \cup U' \) to \( \mathbb{C} \) analytic where \( G|_U = g \).

By the identity principle, \( G \equiv 0 \), so \( g \equiv 0 \) on \( U \).

By the identity principle again, \( g \equiv 0 \).

Since \( h \) is invertible, \( g \circ h^{-1} = f \equiv 0 \).

8. Show that every elliptic function \( f \) of order \( m \) has \( m \) zeros in its fundamental parallelogram.

Since \( f \) is elliptic, so is \( f'/f \), with the same fundamental parallelogram.

Let \( \gamma \) be the path around the fundamental parallelogram, \( Z(f, \gamma) \) be the number of zeros of \( f \) inside \( \gamma \), and \( P(f, \gamma) \) be the number of poles of \( f \) inside \( \gamma \) (both including multiplicity).
\[ Z(f, \gamma) - P(f, \gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} \]

\[ Z(f, \gamma) - m = \frac{1}{2\pi i} \left( \int_{a}^{a+\omega_{1}} \frac{f'}{f} + \int_{a+\omega_{1}}^{a+\omega_{1}+\omega_{2}} \frac{f'}{f} + \int_{a+\omega_{1}+\omega_{2}}^{a+\omega_{2}} \frac{f'}{f} + \int_{a+\omega_{2}}^{a} \frac{f'}{f} \right) \]

\[ Z(f, \gamma) = \frac{1}{2\pi i} \left( \int_{a}^{a+\omega_{1}} \frac{f'}{f} + \int_{a+\omega_{1}}^{a+\omega_{1}+\omega_{2}} \frac{f'}{f} - \int_{a+\omega_{1}+\omega_{2}}^{a+\omega_{2}+\omega_{1}} \frac{f'}{f} - \int_{a+\omega_{2}+\omega_{1}}^{a+\omega_{2}} \frac{f'}{f} \right) + m \]

So, \( f \) has \( m \) zeros in the fundamental parallelogram.