1. Describe all the values of \((-1)^i\), where \(i = \sqrt{-1}\).

\[
(-1)^i = e^{i \log(-1)}
= e^{i (\ln|\sqrt{-1}| + i \arg(\sqrt{-1}))}
= e^{i (i \arg(-1))}
= e^{i \arg(-1)}
= e^{- \arg(-1)}
= e^{(2n+1)\pi}, \text{ with } n \in \mathbb{Z}
\]

2. Write three terms of the Laurent expansion of \(f(z) = \frac{1}{z(z-1)(z-2)}\) in the annulus \(1 < |z| < 2\).

First we will do a partial fraction decomposition of \(\frac{1}{z(z-1)(z-2)}\).

\[
\frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2} = \frac{1}{z(z-1)(z-2)}
\]

\[
A(z-1)(z-2) + Bz(z-2) + Cz(z-1) = 1
\]

\[
z^2(A + B + C) + z(-3A - 2B - C) + 2A = 1
\]
\[ \begin{align*}
2A &= 1 & A + B + C &= 0 & -3A - 2B - C &= 0 \\
A &= \frac{1}{2} & \frac{1}{2} + B + C &= 0 & -\frac{3}{2} - 2B + \frac{1}{2} + B &= 0 \\
B + C &= -\frac{1}{2} & -1 - B &= 0 \\
C &= -\frac{1}{2} - B & B &= -1 \\
C &= \frac{1}{2} &
\end{align*} \]

\[
f(z) = \frac{1}{z(z - 1)(z - 2)}
= \left(\frac{1}{2}\right) \frac{1}{z} - \frac{1}{z - 1} + \left(\frac{1}{2}\right) \frac{1}{z - 2}
= \left(\frac{1}{2}\right) \frac{1}{z} - \left(\frac{1}{z}\right) \frac{1}{1 - 1/z} - \left(\frac{1}{4}\right) \frac{1}{1 - z/2}
= \left(\frac{1}{2}\right) \frac{1}{z} - \left(\frac{1}{z}\right) \left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right) - \left(\frac{1}{4}\right) \left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \ldots\right)
\]

Three terms of the Laurent expansion are then \(-\frac{1}{4}, -\frac{z}{8}, \text{and } -\frac{z^2}{16}\).

3. Give an explicit conformal mapping from the half-disk \( \{ z \mid |z| < 1, \text{Re}(z) > 0 \} \) to the unit disk \( \{ z \mid |z| < 1 \} \).

Let \( f(z) = \frac{z+1}{1-z} \).

If \( x + iy \) is in the upper half disk, then \( x^2 + y^2 < 1, y > 0 \).

\[
f(x + iy) = \frac{1 + x + iy}{1 - x - iy} = \frac{1 - x^2 - y^2 + 2iy}{(1-x)^2 + y^2}.
\]

\[
\text{Re}(f(x + iy)) = \frac{1 - x^2 - y^2}{(1-x)^2 + y^2} > 0.
\]

\[
\text{Im}(f(x + iy)) = \frac{2y}{(1-x)^2 + y^2} > 0.
\]

So \( f(x + iy) \) is in the first quadrant.

\[
f^{-1}(z) = \frac{z-1}{z+1}.
\]

If \( x + iy \) is in the first quadrant, \( f^{-1}(x + iy) = \frac{x-1+iy}{x+1+iy} = \frac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2} \).

\[
\text{Im}(f^{-1}(x + iy)) = \frac{2y}{(x+1)^2 + y^2} > 0.
\]
\[ |f^{-1}(x + iy)| = \frac{|x-1+iy|}{|x+1+iy|} = \frac{(x-1)^2+y^2}{(x+1)+y^2} < 1. \]

So \( f^{-1}(x + iy) \) is in the upper half disk.

So \( f \) is a conformal mapping from the half-disk to the first quadrant.

Let \( g(z) = z^2 \), a conformal mapping from the first quadrant to the upper half-plane.

Let \( h(z) = \frac{i\bar{z}}{z+1} \), a conformal mapping from the upper half-plane to the unit disk.

Then \( h \circ g \circ f \) is a conformal mapping from the half-disk to the unit disk.

4. Determine the radius of convergence of the power series for \( \sqrt{z} \) expanded at \(-4 + 3i\)?

Let \( f(z) = e^{\frac{1}{2} \ln |z| + i \text{Arg}(z))} \) where \( 0 \leq \text{Arg}(z) < 2\pi \).

\[ f(z)^2 = \left( e^{\frac{1}{2} \ln |z| + i \text{Arg}(z))} \right)^2 = e^{\ln |z| + i \text{Arg}(z))} = z. \]

So \( f(z) \) is a square root of \( z \).

\( f(z) \) is holomorphic everywhere except when \( z \geq 0 \).

So, the radius of convergence for the power series of \( f \) is at least \( | -4 + 3i - 0 | = 5 \).

Let \( g \) be a function with \( g(z)^2 = z \).

Let \( c_0 + c_1 z + ... \) be the power series of \( g \) expanded at \(-4 + 3i\).

Then \( (c_0 + c_1 z + ...)^2 = c_0^2 + 2c_0 c_1 z + ... = z \) in the radius of convergence.

If \( g \) has a radius of convergence greater than 5, then \( c_0^2 + 2c_0 c_1 z + ... = z \) at 0.

This means \( c_0 = 0 \).

But if \( c_0 = 0 \), then the linear term of \( c_0^2 + 2c_0 c_1 z + ... \) is 0, and the series cannot converge to \( z \) in a neighborhood around 0.

So, \( g \) must not have a radius of convergence larger than 5.

The radius of convergence of the power series for \( \sqrt{z} \) expanded at \(-4 + 3i \) is 5.

5. Evaluate \( \int_{-4}^{\infty} \frac{e^{\xi x} dx}{1+x^2} \) for real \( \xi \).

First assume \( \xi \geq 0 \).

Let \( \gamma_R \) be the curve that travels from \(-R \) to \( R \) along the real axis and then from \( R \) to \(-R \) via the upper semicircle.

\[
\int_{\gamma_R} \frac{e^{\xi x} dx}{1+x^2} = 2\pi i \text{ Res}_{x=i} \frac{e^{\xi x}}{1+x^2}
\]

\[
= 2\pi i \frac{e^{\xi i}}{2i}
\]

\[
= \pi e^{-\xi}
\]
\[
\left| \int_{C_R} \frac{e^{i\xi x}}{1 + x^2} \right| \leq \int_{C_R} \frac{|e^{i\xi x}|}{1 + x^2} \, dx \leq \int_{C_R} \frac{|e^{i\xi (a+bi)}|}{|R^2 - 1|} \, dx \text{ where } x = a + bi \\
= \int_{C_R} \frac{|e^{i\xi a}| |e^{i\xi b}|}{|R^2 - 1|} \, dx \\
= \int_{C_R} \frac{|e^{-\xi b}|}{|R^2 - 1|} \, dx \\
\leq \int_{C_R} \frac{1}{|R^2 - 1|} \, dx \text{ as } \xi, b \geq 0 \\
= \frac{2\pi R}{|R^2 - 1|} \\
\to 0 \text{ as } R \to \infty
\]

\[
\lim_{R \to \infty} \int_{\gamma_R} \frac{e^{i\xi x}}{1 + x^2} = \lim_{R \to \infty} \left( \int_{-R}^{R} \frac{e^{i\xi x}}{1 + x^2} + \int_{C_R} \frac{e^{i\xi x}}{1 + x^2} \right) \\
\pi e^{-\xi} = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{i\xi x}}{1 + x^2} + \lim_{R \to \infty} \int_{C_R} \frac{e^{i\xi x}}{1 + x^2} \\
\pi e^{-\xi} = \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{1 + x^2}
\]

Now, assume \(\xi < 0\).

Let \(\gamma_R\) be the curve that travels from \(-R\) to \(R\) along the real axis and then from \(R\) to \(-R\) via the lower semicircle.

\[
\int_{\gamma_R} \frac{e^{i\xi x}}{1 + x^2} = - \int_{-\gamma_R} \frac{e^{i\xi x}}{1 + x^2} \\
= -2\pi i \text{ Res}_{x = -i} \frac{e^{i\xi x}}{1 + x^2} \\
= -2\pi i \frac{e^{i\xi (-i)}}{-2i} \\
= \pi e^{\xi}
\]
\[
\left| \int_{C_R} \frac{e^{i\xi x}}{1 + x^2} \, dx \right| \leq \int_{C_R} \frac{|e^{i\xi x}|}{1 + x^2} \, dx
\]

where \( x = a + bi \)

\[
= \int_{C_R} \frac{|e^{i\xi a}||e^{i\xi b}|}{|R^2 - 1|} \, dx
\]

\[
= \int_{C_R} \frac{|e^{-\xi b}|}{|R^2 - 1|} \, dx
\]

\[
\leq \int_{C_R} \frac{1}{|R^2 - 1|} \, dx \quad \text{as} \quad \xi, b \leq 0
\]

\[
= \frac{2\pi R}{|R^2 - 1|}
\]

\[
\rightarrow 0 \quad \text{as} \quad R \rightarrow \infty
\]

\[
\lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{e^{i\xi x}}{1 + x^2} \, dx = \lim_{R \rightarrow \infty} \left( \int_{-R}^{R} \frac{e^{i\xi x}}{1 + x^2} \, dx + \int_{C_R} \frac{e^{i\xi x}}{1 + x^2} \, dx \right)
\]

\[
\lim_{R \rightarrow \infty} \pi e^{\xi} = \lim_{R \rightarrow \infty} \int_{-R}^{R} \frac{e^{i\xi x}}{1 + x^2} \, dx + \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{i\xi x}}{1 + x^2} \, dx
\]

\[
\pi e^{\xi} = \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{1 + x^2} \, dx
\]

So, for any real \( \xi \), \( \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{1 + x^2} \, dx = \pi e^{-|\xi|} \).

6. Show that a holomorphic functions \( f \) on \( \mathbb{C} \) satisfying \( |f(z)| \leq \sqrt{1 + |z|} \) for all \( z \in \mathbb{C} \) is a constant.

Let \( \sum_{k=0}^{\infty} c_k z^k \) be the power series expansion for \( f \) around 0.
\[ |c_k| = \left| \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z^{k+1}} \, dz \right| \] where \( C_R \) is the circle of radius \( R \) centered at 0
\[ \leq \frac{1}{2\pi i} \int_{C_R} \left| \frac{f(z)}{z^{k+1}} \right| \, dz \]
\[ \leq \frac{1}{2\pi i} \int_{C_R} \frac{\sqrt{1+R}}{R^{k+1}} \, dz \]
\[ = \frac{2\pi R\sqrt{1+R}}{2\pi iR^{k+1}} \]
\[ = \frac{\sqrt{1+R}}{iR^k} \]
\[ \to 0 \text{ as } R \to \infty \text{ if } k > 0 \]

So, \( c_k = 0 \ \forall k > 0 \).

7. Show that \( 4z^5 - z + 2 \) has all its zeros in the unit disk.

Let \( f(z) = 4z^5 - z + 2, g(z) = 4z^5 \).

On the unit circle, we get the following inequality:

\[ |f(z) - g(z)| = |4z^5 - z + 2 - 4z^5| \]
\[ = |-z + 2| \]
\[ \leq 3 \]
\[ < 4 \]
\[ = |4z^5| \]
\[ = |g(z)| \]
\[ \leq |f(z)| + |g(z)| \]

So, by Rouché’s Theorem, \( f \) and \( g \) have the same number of zeros inside the unit circle.

\( g \) has 5 zeros inside the unit circle (counting multiplicity), so \( f \) does as well.

\( f \) has 5 zeros total, so all zeros of \( f \) are inside the unit circle.

8. Show that there is a holomorphic function \( f(z) \) on a neighborhood of 0 so that \( f(z)^2 = \frac{\sin z}{z} \) and determine the radius of convergence of the power series at 0.

\[ \sin(z) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \]
So, $g(z) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots = \frac{\sin(z)}{z}$ on $\mathbb{C} \setminus \{0\}$.

Define $f(z) = e^{\frac{1}{2}(\ln|g(z)| + i\text{Arg}(g(z)))}$ where $-\pi < \text{Arg}(z) \leq \pi$.

$f(0) = e^{\frac{1}{2}(\ln|g(0)| + i\text{Arg}(g(0)))} = e^{\frac{1}{2}(\ln 1 + i\text{Arg}(1))} = e^0 = 1$.

$f$ is holomorphic when $g(z) \not\leq 0$.

The zeros of $g$ are the same as the zeros of $\sin(z)/z$, $\{n\pi \mid n \neq 0\}$.

The closest zero to the point 0 is the zero at $\pi$ (or $-\pi$).

The largest disc where $f$ is holomorphic is the disc with radius $\pi$. Note there is no closer point were $g$ is negative, because in order for $g$ to be negative at a point closer to 0, there must be a zero closer to 0, as $g(0) = 1$.

So, the radius of convergence for the power series of $f$ at 0 is $\pi$.

9. Describe all complex-valued harmonic functions on the annulus $1 < |z| < 2$ which extend continuously to the circle $|z| = 2$ and take value 0 on that circle.

We know that all complex-valued harmonic functions on an annulus are of the form $a_0 + b_0 \log |z| + \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k z^k + b_k \bar{z}^k$.

If $|z| = 2$, $z = 2e^{i\theta}$ for some $\theta$.

$0 = a_0 + b_0 \ln 2 + \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k 2^k e^{ik\theta} + b_k 2^k e^{-ik\theta} = a_0 + b_0 \ln 2 + \sum_{k \in \mathbb{Z} \setminus \{0\}} (a_k 2^k + b_{-k} 2^{-k}) e^{ik\theta}$

$\sum_{k \in \mathbb{Z} \setminus \{0\}} (a_k 2^k + b_{-k} 2^{-k}) e^{ik\theta}$ is a constant, so $\sum_{k \in \mathbb{Z} \setminus \{0\}} ik(a_k 2^k + b_{-k} 2^{-k}) e^{ik\theta} = 0$.

But since $e^{ik\theta}$ is an orthogonal system, this is only possible if $ik(a_k 2^k + b_{-k} 2^{-k}) = 0 \forall k$.

Then $-a_k 2^k = b_{-k} 2^{-k}$.

$-a_{-k} 2^{-2k} = b_k$.

Also, $\frac{-a_0}{\ln 2} = b_0$.

So, $f(z) = a_0 - \frac{a_0}{\ln 2} \log |z| + \sum_{k \in \mathbb{Z} \setminus \{0\}} a_k z^k - a_{-k} 2^{-2k} \bar{z}^k$. 