Problem 1. Assume that $S$ is a Lebesgue measurable subset of $\mathbb{R}$ and that $S$ has positive measure. Show that existence of two distinct points $x \in S$ and $y \in S$ such that $x - y$ is rational.

Proof. Since $S$ has positive measure, we know that for some interval $[n, n + 1]$, $S \cap [n, n + 1]$ has positive measure for some $n \in \mathbb{N}$. Let $\{q_k\}$ be some enumeration of the rationals from $[-1, 1]$. Let $S_k = (S \cap [n, n + 1]) + q_k$ - a rational shift of $S \cap [n, n + 1]$.

We will assume for contradiction that the $S_k$’s are disjoint. If they were not disjoint then there would exist a $k, \ell$ such that $S_k \cap S_\ell \neq \emptyset$. Let $z \in S_k \cap S_\ell$. Then $z = x + q_k$, where $x \in S \cap [n, n + 1]$ and $z = y + q_\ell$ where $y \in S \cap [n, n + 1]$. Then

$$0 = z - z = x + q_k - y - q_\ell \implies x - y = q_k - q_\ell \in \mathbb{Q}.$$ 

Since they are shifts, $m(S \cap [n, n + 1]) = m(S_k)$. Note that $S \cap [n, n + 1] \subset \bigcup_{k=1}^{\infty} S_k \subset [n-1, n+2]$.

Now we know that

$$0 < m(S \cap [n, n + 1]) \leq m\left(\bigcup_{k=1}^{\infty} S_k\right)$$

$$= \sum_{k=1}^{\infty} m(S_k) \quad \text{since } S_k\text{'s are disjoint}$$

$$= \sum_{k=1}^{\infty} m(S \cap [n, n + 1]) \leq 3$$

but $\sum_{k=1}^{\infty} m(S \cap [n, n + 1])$ is infinite so this is a contradiction. Therefore, there must exist two distinct points whose difference belongs in $\mathbb{Q}$. \qed
Problem 2. Prove or disprove:

a. There exists a set $K \subset [0, 1]$ such that $K$ is closed, uncountable, and has Lebesgue measure zero.

b. There exists set $K \subset [0, 1]$ such that $K$ is closed, nowhere dense, and has positive Lebesgue measure.

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a. True.

Define $K := C$ - the standard middle-thirds Cantor Set. By definition, $C \subset [0, 1]$. $C$ is the complement of the union of open sets, which is open. Thus it is closed. Closed sets are measurable.

$$m(C) = m([0, 1]) - m(C^c) = 1 - \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^n = 1 - \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}}\right) = 1 - \frac{1}{3} \left(\frac{1}{\frac{1}{2}}\right) = 0$$

Recall the Cantor-Lebesgue function which is a continuous bijection from $[0, 1]$ to $[0, 1]$ which is only increasing on the Cantor set. Because the function is a bijection, this puts the Cantor set into a bijective correspondence with $[0, 1]$. Therefore, the Cantor set is uncountable.

Alternative: Every element of $[0, 1]$ has a ternary expansion. The elements of the Cantor set are exactly those with only 0s and 2s in a ternary expansion. Represent all of the Cantor set elements in their ternary expansions with only 0s and 2s and replace the 2s with 1s. Then we get all possible elements of $[0, 1]$ as represented in their binary expansions. So we have a surjection from the Cantor set to $[0, 1]$, and the Cantor set is uncountable.

b. True.

Define $K := C^*$ - the Fat-Cantor Set (Smith-Volterra Cantor Set). By definition, $C^* \subset [0, 1]$. $C^*$ is the complement of the union of open sets, which is open. Thus it is closed. Closed sets are measurable.

$$m(C^*) = m([0, 1]) - m([0, 1] \setminus C^*) = 1 - \sum_{n=0}^{\infty} \frac{1}{4} \left(\frac{2}{4}\right)^n = 1 - \frac{1}{4} \left(\frac{1}{1 - \frac{1}{2}}\right) = 1 - \frac{1}{4} \left(\frac{1}{\frac{1}{2}}\right) = \frac{1}{2}$$

Now we see that since $C^*$ is closed, and thus $\text{int}(C^*) = \text{int}(C^c)$. Since $C^*$ contains no intervals, we know that $\emptyset = \text{int}(C^*) = \text{int}(C^c)$. Thus, $C^*$ is a nowhere dense set by definition.
**Problem 3.** Prove or disprove:

a. Every locally compact metric space is complete.

b. Every locally compact inner product space is finite dimensional.

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### a. False.

Define \( x_n := \frac{1}{n}, \forall n \in \mathbb{N} \). Let \( X := (0,1) \subset \mathbb{R} \). \( X \) is locally compact as a consequence of the Heine-Borel Theorem. We want to show that \( x_n \) is Cauchy convergent with \( x_n \in X \), but does not converge in \( X \). For a sequence to be Cauchy convergent, \( \forall \varepsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that \( \forall n, m \geq N, |x_n - x_m| < \varepsilon \). Assume without loss of generality that \( n \geq m \). So, we have:

\[
|x_n - x_m| = \frac{1}{n} - \frac{1}{m} = \frac{m - n}{nm} < \frac{m}{nm} = \frac{1}{n} < \varepsilon
\]

Take \( N = \frac{1}{\varepsilon} \), then \( n, m \geq \frac{1}{\varepsilon}, \Rightarrow |x_n - x_m| < \varepsilon \). Thus we can see that \( x_n \) is Cauchy convergent, but it converges to the value 0, which is not in \( X \).

### b. True.

*Proof.* Assume for contradiction that the inner product space, \( X \), is infinite dimensional. Since the space is locally compact, around the origin, \( O \), specifically there is a compact neighborhood \( U \). Choose \( \{e_n\}_1^{\infty} \), an orthogonal sequence with norm 1. So \( \{re_n\}_1^{\infty} \subset B_r(O) \). By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence such that \( \{re_{n_k}\}_1^{\infty} \subset B_r(O) \). In particular, this sequence is Cauchy. So we see:

\[
||re_{n_k} - re_{n_l}||^2 = ||re_{n_k}||^2 + 2\langle re_{n_k}, re_{n_l} \rangle + ||re_{n_l}||^2 = 2r
\]

and so \( ||re_{n_k} - re_{n_l}|| = r\sqrt{2} \) and thus it is not Cauchy - which is a contradiction. So \( X \) must be finite dimensional. \( \square \)
Problem 4. a. State the Riemann-Lebesgue Lemma for functions in $L^1([0,1])$.

b. Using your statement in part (a), prove the following statement. If $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable periodic functions with period 1, then

$$n \int_0^1 f(x) \sin(2\pi nx) dx \to 0 \text{ as } n \to \infty$$

where $n$ is an integer.

c. Is the statement in part (b) true if $f$ is not periodic? Why or why not?

a. Let $f \in L^1([0,1])$, then the Riemann-Lebesgue Lemma states

$$\hat{f}(n) = \int_0^1 f(x) \cos(nx) dx \to 0 \text{ as } n \to \infty$$

b. Proof. Consider the function $f'(x)$. Since $f'(x)$ is given to be continuous, we know that on the compact set $[0,1]$, $f'(x)$ is bounded. Thus we have that $f'(x) \in L^1([0,1])$. Now by the Riemann-Lebesgue Lemma

$$\int_0^1 f'(x) \cos(2\pi nx) dx \to 0 \text{ as } 2\pi n \to \infty$$

$$\implies \int_0^1 f'(x) \cos(2\pi nx) dx \to 0 \text{ as } n \to \infty$$

Now we integrate by parts, and get

$$\int_0^1 f'(x) \cos(2\pi nx) dx = f(x) \cos(2\pi nx) \bigg|_0^1 + 2\pi n \int_0^1 f(x) \sin(2\pi nx) dx$$

since $f(x)$ is of period 1. Thus we now have that by the Riemann-Lebesgue Lemma

$$\lim_{n \to \infty} 2\pi n \int_0^1 f(x) \sin(2\pi nx) dx = 0$$

$$\implies \lim_{n \to \infty} n \int_0^1 f(x) \sin(2\pi nx) dx = 0$$

c. Solution. The statement in part (b) is false if $f$ is not periodic. Consider $f(x) = x$. Then we see this is not of period 1 on $[0,1]$ but is still in $L^1([0,1])$. We then notice that when we do integration by parts, the first term does not disappear, and in fact we pick up a 1. Thus we cannot make the same conclusion.
Problem 5. Let \( f \in L^1(\mathbb{R}) \), let \( f > 0 \), and let \( \hat{f} \) be the Fourier transform of \( f \). Compute \( \sup_{x \in \mathbb{R}} |\hat{f}(x)| \), and show that the supremum is achieved at exactly one point.

Proof. We claim that
\[
\sup_{x \in \mathbb{R}} |\hat{f}(x)| = ||f||_1.
\]

We can see that the supremum exists since
\[
\sup_{x \in \mathbb{R}} |\hat{f}(x)| = \sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} f(t)e^{-2\pi i xt} dt \right| \\
\leq \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |f(t)| \cdot |e^{-2\pi i x}| dt \\
= \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |f(t)| dt \\
= \int_{\mathbb{R}} |f(t)| dt = ||f||_1
\]

We also see that when \( x = 0, \forall f \in L^1(\mathbb{R}) \), we achieve \( ||f||_1 \):
\[
\sup_{x \in \mathbb{R}} |\hat{f}(0)| = \sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} f(t)e^{-2\pi i 0t} dt \right| = \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |f(t)| dt = \int_{\mathbb{R}} |f(t)| dt = ||f||_1
\]
since \( f > 0 \).

Now we want to show this holds only when \( x = 0 \).

So, if \( |\hat{f}(x)| = ||f||_1 \), then since \( f > 0 \), we have
\[
|\hat{f}(x)|^2 = ||f||_1^2 \\
\hat{f}(x)f(x) = \int_{\mathbb{R}} f(t)dt \cdot \int_{\mathbb{R}} f(s)ds \\
\int_{\mathbb{R}} f(t)e^{-2\pi i xt} dt \cdot \int_{\mathbb{R}} f(s)e^{2\pi i xs} ds = \int_{\mathbb{R}} f(t)dt \cdot \int_{\mathbb{R}} f(s)ds \\
\int_{\mathbb{R}} f(t)f(s)e^{2\pi i (s-t)} dtds = \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)f(s)dtds \\
\int_{\mathbb{R}} \int_{\mathbb{R}} f(t)f(s) \left(e^{2\pi i (s-t)} - 1 \right) dtds = 0 \\
Re \left( \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)f(s) \left(e^{2\pi i (s-t)} - 1 \right) dtds \right) = Re \left( 0 \right) \\
\int_{\mathbb{R}} \int_{\mathbb{R}} f(t)f(s) \left(cos(2\pi x(s-t)) - 1 \right) dtds = 0
So, \( \cos(2\pi x(s - t)) \in [-1, 1] \implies (\cos(2\pi x(s - t)) - 1) \in [-2, 0] \). Thus since \( f(t), f(s) > 0 \) and \( (\cos(2\pi x(s - t)) - 1) \leq 0 \), this tells us that the product is non-positive. Since this quantity integrates to 0, this means that \( (\cos(2\pi x(s - t)) - 1) \) must be 0 almost everywhere. This will hold iff \( s - t \in \mathbb{Z} \) and \( x \in \mathbb{Z} \), or \( x = 0 \).

Consider the intervals \( t \in [0, \frac{1}{4}] \) and \( s \in [\frac{1}{2}, \frac{3}{4}] \). Then we see that \( s - t \in [\frac{1}{4}, \frac{3}{4}] \). Here, \( s - t \notin \mathbb{Z} \). This is also a set of positive measure. Thus, the only way the integral will be 0 here is if \( x = 0 \). So we have ruled out the case \( s - t, x \in \mathbb{Z} \), and \( x \) must be 0 for this to hold. Therefore, the supremum is achieved at only one point, \( x = 0 \). \( \square \)
**Problem 6.** Let $K$ be a compact subset of $\mathbb{R}$, and suppose that $f : K \rightarrow \mathbb{R}$ and $f_n : K \rightarrow \mathbb{R}$, $n = 1, 2, \ldots$, are continuous. Suppose that, for every $x \in K$, $f_{n+1}(x) \leq f_n(x)$, $n = 1, 2, \ldots$, and $\lim_{n \to \infty} f_n(x) = f(x)$.

a. Show that $f_n(x) \to f(x)$ uniformly on $K$ as $n \to \infty$.

b. Give an example to show that the compactness of $K$ is necessary.

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**Proof.** Fix $\varepsilon > 0$. Let $g_n(x) = f_n(x) - f(x)$. Then $g_n(x)$ is positive because $f_n(x) \geq f(x)$ for every $x$ and $g_n(x)$ is continuous for every $n$ since $f$ and $f_n$, $n = 1, 2, \ldots$, are all continuous. Let $E_m = \{ x : |g_m(x)| < \varepsilon \}$. Note that $E_m$ is open because it is the preimage of $(-\varepsilon, \varepsilon)$ under $g_m$, which is continuous. Also notice that because $f_n(x)$ converges pointwise to $f(x)$, every $x$ is in $E_m$ for some $m \in \mathbb{N}$. Thus $K = \bigcup_{m=1}^{\infty} E_m$. $K$ is compact, so $\exists$ a finite cover $E_{i_1}, \ldots, E_{i_N}$. Since the $g_n$s decrease, $E_1 \subseteq E_2 \subseteq \ldots$. This means for any $x \in K$, $x \in E_{m_N}$ so $|g_{m_N}(x)| = |f_{m_N}(x) - f(x)| < \varepsilon$. Thus $f_n \to f$ uniformly on $K$ as $n \to \infty$. \qed

b. Consider the sequence of functions $f_n(x) = x^n$ on the open interval $(0, 1)$. Then $f_n(x) \to f(x) = 0$ as $n \to \infty$ and $f$, and $f_n$, $n = 1, 2, \ldots$ are all continuous on $(0, 1)$. However, $f_n(x)$ does not converge uniformly to 0 because for any $n$, there is an $x \in (0, 1)$ such that $x^n > \frac{1}{2}$. Thus the compactness of $K$ is necessary.
Problem 7. a. State Fubini’s Theorem for $L^1$ functions on measure spaces. Be sure to state the hypotheses and conclusions fully and precisely.

b. Show that
$$\int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx = \frac{\pi}{4}.$$ 
Hint: \( \frac{d}{dy} \left( \frac{y}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \)

c. Show that
$$\int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) \, dy = -\int_0^1 \left( \int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2} \, dy \right) \, dx.$$ 

d. Explain why parts (b) and (c) do not provide a counterexample to Fubini’s Theorem.

a. **Theorem (Fubini’s Theorem).** Let $X$ and $Y$ be measure spaces. Let $X \times Y$ denote the maximal product measure. If $f(x, y)$ is integrable on $X \times Y$, then

(i) For almost every $x \in X$, the slice $f^x(y)$ is integrable on $Y$.
(ii) For almost every $y \in Y$, the slice $f^y(x)$ is integrable on $X$.
(iii) $\int_X \int_Y f(x, y) \, dy \, dx = \int_Y \int_X f(x, y) \, dx \, dy = \int_{X \times Y} f(x, y) \, d(x, y)$.

b. $\int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx = \int_0^1 \left( \int_0^1 \frac{y}{x^2 + y^2} \, dy \right) \, dx$

$$= \int_0^1 \left. \frac{1}{x^2 + 1} \right|_0^1 \, dx = \arctan(x) \bigg|_0^1 = \frac{\pi}{4}$$

c. From the hint, we realize that $\frac{d}{dx} \left( \frac{x}{x^2 + y^2} \right) = \frac{y^2 - x^2}{(x^2 + y^2)^2}$. So with this in mind,

$$\int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) \, dy = -\int_0^1 \left( \int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2} \, dy \right) \, dx$$

$$= -\int_0^1 \left. \frac{x}{x^2 + y^2} \right|_0^1 \, dy$$

$$= -\int_0^1 \frac{1}{y^2 + 1} \, dy = -\arctan(y) \bigg|_0^1 = -\frac{\pi}{4}$$
d. Observe,
\[
\int_0^1 \int_0^2 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx \geq \int_0^1 \int_0^x \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx \quad \text{(this is where } x \geq y) \\
= \int_0^1 \frac{y}{x^2 + y^2} \bigg|_0^x \, dx \\
= \int_0^1 \frac{x}{2x^2} \, dx = \frac{1}{2} \int_0^1 \frac{1}{x} \, dx
\]
where we know that \( \int_0^1 \frac{1}{x} \, dx \) does not converge. Thus we see that the integral,
\[
\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \, dx = \infty.
\]
And so, our function is not in \( L^1(\mathbb{R}^2) \) and Fubini’s Theorem does not apply.