Plabic Networks on a Cylinder

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Graduate Student Combinatorics Conference

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Theory of plabic networks in a disk developed by Postnikov in studying connection between totally nonnegative Grassmannian and planar directed networks in a disk
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- Soliton solutions to the KP equation
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- Scattering amplitudes for $\mathcal{N} = 4$ supersymmetric Yang-Mills
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Applications:

- Soliton solutions to the KP equation
- Scattering amplitudes for $\mathcal{N} = 4$ supersymmetric Yang-Mills
- Poisson geometry
Postnikov Diagrams
A plabic graph on a cylinder is a planar undirected graph such that each boundary vertex has degree 1 and each internal vertex is colored black or white.

We can obtain a plabic graph from a Postnikov diagram as follows:

- Place a black vertex in every face oriented counterclockwise and a white vertex in every face oriented clockwise.
- If two oriented faces share a corner, connect the vertices in these two faces.
Postnikov Diagrams and Plabic Graphs

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Perfect Orientation

Definition (Postnikov, 2006)

A perfect orientation of a plabic graph is an orientation of edges such that every black vertex has exactly one edge outgoing and every white vertex has exactly one incoming vertex.
A *perfect orientation* of a plabic graph is an orientation of edges such that every black vertex has exactly one edge outgoing and every white vertex has exactly one incoming vertex.
A trail in a perfectly oriented plabic graph on a cylinder $G$ is a sequence of vertices $v_1, ..., v_{m+1}$ where $v_1, v_{m+1}$ are boundary vertices on different boundary components and for each $i$, either $(v_i, v_{i+1})$ or $(v_{i+1}, v_i)$ is an edge in $G$. 
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oriented left to right
Definition

A directed plabic network on a cylinder is a perfectly oriented plabic graph on a cylinder with a weight $y_f \in \mathbb{R}_{>0}$ associated to each face such that $\prod y_f = 1$ and a specified trail with weight $t$. 
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Path Weights

We can define the weight $wt(P, y, t)$ of any path $P$ is a directed plabic network.
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\[ \frac{1}{y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8} \]
Definition (Gekhtman-Shapiro-Vainshtein, 2012)

Label the boundary vertices of a directed plabic network $b_1, \ldots, b_n$. The boundary measurements for a planar network on a cylinder are

$$M_{ij} := \sum_{\text{paths } P \text{ from } b_i \text{ to } b_j} (-1)^{\text{wind}(C_P)-1} \zeta^{\text{int}(P)} \text{wt}(P, y, t).$$
The following local transformations are moves:

\[
\begin{align*}
&y_0 \leftrightarrow \frac{y_1}{1 + y_0^{-1}} \quad &y_4(1 + y_0) \quad &y_0^{-1} \quad &y_2(1 + y_0) \\
&y_4 \leftrightarrow y_1 \quad &y_0 \leftrightarrow y_2 \quad &y_3 \leftrightarrow y_5 \\
&y_1 \leftrightarrow y_2
\end{align*}
\]
Theorem (Postnikov, 2006)

Applying moves doesn’t change the boundary measurements.
A cylindric $k$-loop plabic graph is a plabic graph on a cylinder that arises from a Postnikov diagram where exactly $k$ of the strands are loops around the cylinder with the same orientation.
Cylindric $k$-loop Plabic Graphs

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Cylindric $k$-loop Plabic Graphs

**Definition**

For a cylindric $k$-loop plabic network, any vertices that are not on one of the strings of vertices defined by the $k$ loops and lie between two of these strings are called *interior vertices.*
Cylindric $k$-loop Plabic Graphs

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**Theorem (C. 2018)**
Any cylindric $k$-loop plabic network can be transformed by moves to one that has no interior vertices.
We define a transformation on weights for cylindric $k$-loop plabic graphs.
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\[
\begin{align*}
  a' &= a_1 a_2 b_1 b_2 b_3 c_1 c_2 c_3 + 1 + c_1 + c_1 c_2 c_3 \\
  b' &= a_1 a_2 b_1 b_2 b_3 c_1 c_2 c_3 + 1 + c_1 + c_1 c_2 c_3
\end{align*}
\]
\[ a'_1 = \frac{a_1 a_2 b_1 b_2 b_3 c_1 c_2 c_3 + 1 + c_1 + c_1 c_2}{a_2 b_1 b_2 b_3 (1 + c_1 + c_1 c_2 + c_1 c_2 c_3)} \]

\[ b'_1 = \frac{b_1 (a_1 a_2 b_1 b_2 b_3 c_1 c_2 + a_1 a_2 b_1 b_2 b_3 c_1 c_2 c_3 + 1 + c_1)}{a_1 a_2 b_1 b_2 b_3 c_1 + a_1 a_2 b_1 b_2 b_3 c_1 c_2 + a_1 a_2 b_1 b_2 b_3 c_1 c_2 c_3 + 1} \]

\[ c'_1 = \frac{a_1 a_2 b_1 b_2 b_3 c_1 (1 + c_1 + c_1 c_2 + c_1 c_2 c_3)}{a_1 a_2 b_1 b_2 b_3 c_1 c_2 + a_1 a_2 b_1 b_2 b_3 c_1 c_2 c_3 + 1 + c_1} \]
Theorem (C. 2018)

Plabic R-matrices have the following properties:

- They preserve the boundary measurements.
- They are involutions.
- They give the only choices of weights on a fixed cylindric 2-loop plabic graph that preserve the boundary measurements.
- They satisfy the braid relation.
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Plabic R-Matrices

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- The plabic R-matrix has an underlying cluster structure.
- We can define our networks with edge weights instead of face and trail weights. This gives an edge weighted version of the plabic R-matrix. In certain cases, this specializes to the geometric R-matrix.