Introduction to Quiver Algebras

Definition: Informally, a quiver is a labeled directed graph with loops and multiple edges allowed.

\[ \begin{array}{c}
\circ & \leftarrow & \circ & \rightarrow & \circ \\
\end{array} \]

are all quivers.

Formally,

Definition: A quiver is a quadruple \((Q_0, Q_1, s, t)\) where \(Q_0\) is the set of vertices, \(Q_1\) is the set of arrows, and \(s, t : Q_1 \rightarrow Q_0\) are maps identifying the source and target, respectively, of arrows in \(Q_1\).

Example:

\[ i \xrightarrow{\alpha} 2 \quad s(\alpha) = 1, \quad t(\alpha) = 2 \]

Definition: \(kQ\), the path algebra or quiver algebra, is a \(k\)-vector space with the set of all paths of length \(\geq 0\) as its basis.

\(Q = \begin{array}{c}
1 & \xrightarrow{\alpha} & 2 \\
\end{array}\)

has \(kQ = \text{span} \{ e_1, e_2, \alpha \} \)

The operation in \(kQ\) is defined by concatenation:

\[ p \cdot x = \begin{cases} 
    0 & \text{if } t(x) = s(p) \\
    p \cdot x & \text{otherwise}
\end{cases} \]

Example:

\[ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \]

\(p \cdot \alpha = p \cdot \alpha \cdot \beta = 0, \quad p \cdot e_2 = \beta, \quad p \cdot e_3 = 0, \text{ etc...} \)

The identity element is the sum of the lazy paths. In the previous example,

\[ \alpha (e_1 + e_2 + e_3) = \alpha \cdot e_1 + \alpha \cdot e_2 + \alpha \cdot e_3 = \alpha \cdot e_1 = \alpha \]

Some examples of quiver algebras:

Example:

\[ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \]

has \(kQ = \text{span} \{ e_1, e_2, \alpha, \beta, \alpha \cdot \beta, \alpha \cdot e_2, \beta \cdot e_1, \beta \cdot e_3, 1 \} \)

Example:

\[ \alpha \xrightarrow{\beta} \gamma \]

\(kQ = \{ 1, \alpha, \beta, \gamma \}\) with empty word \(e_1^2 \in k < t_1, t_2 > \)

\(e_1 \mapsto 1, \quad \alpha \mapsto t_1, \quad \beta \mapsto t_2 \)
Notes: KG not always finite.

Let $I$ be a two-sided ideal of $KQ$. Then $\exists$ a sufficient condition on $KQ$ s.t. $KQ/I$ is finite.

Definition: The two-sided ideal $R_a$, generated by the arrows of $Q$ (i.e., paths of length 1), is called the arrow ideal of $KQ$.

Definition: A two-sided ideal $I$ is admissible if $\exists m \geq 2$ s.t. $R_a^m \subseteq I \subseteq R_a^2$.

Unpacking this definition:
- $R_a^2 = \{\text{all paths of length 2}\}$
- $R_a^m = \{\text{all paths of length } m\}$

$I$ contains paths of length $m$. If $R_a^m \subseteq I$ implies $I$ contains all paths of length $2m$.

But possibly some subset of paths with lengths between 2 and $m-1$.

$I$ is $R_a^2$ ensures $KQ/I$ is connected.

Definition: For admissible $I$, $KQ/I$ is a bound quiver algebra.

EXII $Q=\begin{array}{ccc} 1 & \alpha & 2 \\ \alpha & 3 \\ \beta \end{array}$ $I = \langle \alpha^2, \alpha \beta, \alpha \beta \alpha \rangle$ is admissible (m=3)

any path with length 3,3 contains $\alpha^2$ or $\alpha \beta$, so $R_a^3 \subseteq I$

Clearly $I \subseteq R_a^2$ since its basis elements have length 3

EXII Different relations on the same quiver can give you the same bound quiver algebra.

$Q=\begin{array}{ccc} 1 & \alpha & 2 \\ \alpha & 3 \\ \beta \end{array}$ $I_1 = \langle \alpha \beta, \alpha \beta \alpha \rangle$ $I_2 = \langle \alpha \beta - \beta \rangle$

So $I_1 \neq I_2$, but $KQ/I_1 \cong KQ/I_2$

Quivers can be used to visualize modules. For a quiver $Q$ with bound quiver algebra $A = KQ/I$, we can visualize any $A$-module $M$ as a $K$-linear representation of $(Q, I)$.
Definition: A \( k \)-linear representation \( M \) of \( Q \) is specified by:

(a) associating each point \( a \in Q_0 \) with a \( k \)-vector space \( M_a \)

(b) associating each arrow \( a \in Q_1 \) with a \( k \)-linear map \( P_a : M_{s(a)} \to M_{t(a)} \)

We denote this as \( M = (M_a, P_a) \).

Definition: \( M \) is finite dimensional if each \( M_a \) is finite dimensional.

\[ \begin{array}{c}
\downarrow \quad 1 \quad 2 \quad \text{has representations} \quad k^2 \quad \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \\
\downarrow \quad 0 \\
\end{array} \]

EX: \[ \begin{array}{ccc}
\downarrow & k & \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \\
\downarrow & 0 & \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \\
\end{array} \]

Let \( Q = (Q_0, Q_1, s, t) \) be an unbound quiver. Then for each \( i \in Q_0 \),

Definition: The \( \text{simple representation} \) \( S(i) \) has dimension 1 at vertex \( i \) and 0 at all other vertices. There is one such representation for each \( i \in Q_0 \).

Definition: The \( \text{projective representation} \) \( P(i) \) is constructed as follows:

(a) Let \( P(i)_a \) be the \( k \)-vector space with basis \( \{ \text{all paths from } i \text{ to } j \text{ in } Q \} \)

(b) For each \( j \to l \) in \( Q_1 \), let \( P_a : P(i)_j \to P(i)_l \) be the linear map defined by composing the paths from \( i \) to \( j \) with \( j \to l \).
Definition: The injective representation $I(i)$ is constructed as

(a) $I(i)_j$ is the $k$-vector space with basis $2^j$ paths from $j$ to $i$ in $Q$.

(b) For $j \xrightarrow{e} i$ in $Q$, $I(i)_j \rightarrow I(i)_i$ is the linear map defined on the basis by deleting $j \xrightarrow{e} i$ from paths from $j$ to $i$ and sending other paths to zero.

$$Q = \begin{array}{cccc}
0 & \rightarrow & 0 & \rightarrow \mathbb{K} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K}^2 \\
\downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{K}^2 & \rightarrow \mathbb{K}^2 \\
\end{array}$$

$S(3) \cong \begin{array}{cccc}
0 & \rightarrow & 0 & \rightarrow \mathbb{K} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K} \\
\end{array}$

$P(3) \cong \begin{array}{cccc}
0 & \rightarrow & 0 & \rightarrow \mathbb{K} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K} \\
\end{array}$

$I(3) \cong \begin{array}{cccc}
0 & \rightarrow & 0 & \rightarrow \mathbb{K} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K} \\
\downarrow & & \downarrow & \\
0 & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K} \\
\end{array}$

Example (easy): $i \xrightarrow{1} j \xrightarrow{2} k \xrightarrow{3} l \xrightarrow{4} m$

Example (more complicated):

$$Q = \begin{array}{cccc}
1 & \rightarrow & 2 & \rightarrow 3 & \rightarrow 4 \\
\Rightarrow & & \downarrow & \downarrow & \\
1 & \rightarrow & 2 & \rightarrow 3 & \rightarrow 4 \\
\end{array}$$

$$P(4) \cong \begin{array}{cccc}
\mathbb{K} & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K}^2 \\
\Rightarrow & & \downarrow & \downarrow & \\
\mathbb{K} & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K}^2 \\
\Rightarrow & & \downarrow & \downarrow & \\
\mathbb{K} & \rightarrow & \mathbb{K} & \rightarrow \mathbb{K}^2 \\
\end{array}$$

Let $M$ be a right $A$-module. Then:

Definition: A module $M$ is called a rigid $A$-module. Then the radical of $M$, $\text{rad}(M)$, is the intersection of all maximal submodules of $M$. Here, $\text{rad}(\mathbb{K}G/I) = R$ if $R$ is a nontrivial $\mathbb{K}$-subspace of $Q$.

$\text{rad}^k(\mathbb{K}G/I) = R^k$ if $R$ is the $k$-th power of $\mathbb{K}$ in $Q$.

Note: all arrows in $\text{Soc}(M)$ act by 0.

Definition: $\text{top}(M) = M/\text{rad}(M)$
Let $M = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (from $G = \begin{array}{ccc} 1 & 2 & 3 \end{array}$).

**Definition:** $\text{soc}(M) = \text{N} = (N_a, \Psi_a)$ with $N_a = M_a$ if $a$ is a sink.

$N_a = \bigcap_{a: b \rightarrow a} \ker(\Psi_b: M_b \rightarrow M_a)$ if $a$ is a sink.

$\Psi_a = \Psi_a|_{N_a} = 0$ for every arrow $x$ of the source $a$.

**Ex:** $\text{soc}(M) = \begin{array}{ccc} 0 \\ 0 \\ 0 \\ 0 \end{array}$

**Definition:** $\text{rad}(M) = \text{J} = (J_a, \gamma_a)$ with $J_a = \sum_{a: b \rightarrow a} \text{Im}(\Psi_b: M_b \rightarrow M_a)$

$\gamma_a = \Psi_a|_{J_a}$ for every arrow $x$ of source $a$.

**Ex:** $\text{rad}(M) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

**Definition:** $\text{top}(M) = \text{L} = (L_a, \Lambda_a)$ with $L_a = M_a$ if $a$ is a source.

$L_a = \sum_{a: b \rightarrow a} \text{coker}(\Psi_b: M_b \rightarrow M_a)$ if $a$ is a source.

$\Psi_a = 0$ for every arrow $x$ of source $a$.

**Ex:** $\text{top}(M) = \begin{array}{ccc} 0 \\ 0 \\ 0 \\ 0 \end{array}$.