$k$: algebraically closed field.

$\Lambda$: finite-dimensional $k$-algebra. (Note: this means a ring hom $k \to Z(\Lambda)$. It is not a $\mathbb{C}$-algebra)

Study the category of finite-dimensional (or $k$) $\boxed{\text{left}}$ $\Lambda$-modules. $\Lambda$-mod ($\Lambda$-Mod would be all modules.) (pretty much all finiteness conditions coincide here) Every module is fin-dim.

**Schur's lemma:** $S, T$ simple $\Lambda$-modules (0≠S with no proper submodules)

(1) $\mathrm{Hom}_\Lambda(S,T) = 0$

(2) $\mathrm{End}_\Lambda(S) \cong k$

**Thm (Artin-Wedderburn):** TFAE:

(1) Every short exact sequence splits in $\Lambda$-mod

(2) Every $\Lambda$-module is projective.

(3) ——————— is injective.

(4) ——————— is a finite (direct) sum of simple modules.

(5) $\Lambda \cong \mathrm{Mat}_{n_1}(k) \times \dots \times \mathrm{Mat}_{n_r}(k)$

I will add more!

**Pf:**

(4) ⇒ (5): Write ${}_\Lambda\Lambda \cong S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$ where $S_i \not\cong S_j$ if $i \neq j$

$\Lambda$ as a module over itself

$$\Lambda^{op} \cong \mathrm{End}_\Lambda({}_\Lambda\Lambda) \cong \mathrm{End}_\Lambda(S_1^{n_1} \oplus \dots \oplus S_r^{n_r})$$

$$\overset{\text{Schur (1)}}{\cong} \mathrm{End}_\Lambda(S_1^{n_1}) \times \dots \times \mathrm{End}_\Lambda(S_r^{n_r})$$

$$\cong \mathrm{Mat}_{n_1}(\mathrm{End}_\Lambda(S_1)) \times \dots \times \mathrm{Mat}_{n_r}(\mathrm{End}_\Lambda S_r)$$

$$\overset{\text{Schur (2)}}{\cong} \mathrm{Mat}_{n_1}(k) \times \dots \times \mathrm{Mat}_{n_r}(k)$$

**Note:** Here, $r$ is the number of noniso. simples.

**Rk (Cartan's criterion):** If char $k = 0$ we can also add

(6) The bilinear form $\Lambda \times \Lambda \to k$, $(x,y) \mapsto \mathrm{tr}(\rho(xy))$ is non-degenerate

as an equ. condition above

If $\Lambda$ satisfies any of the equ. conditions, it is called semisimple.

Defn: A $\Lambda$-module $M$ is called semisimple if it is a (direct) sum of simples.

Over a non-semisimple $\Lambda$, there typically are lots of non-semisimple $\Lambda$-modules.

Approximate:

Defn: $\mathrm{soc}(M) := \sum_{\substack{N \leq M \\ N\text{-simple}}} N > 0$ $\quad$ $\mathrm{rad}(M) := \bigcap_{\substack{N \leq M \\ M/N \text{ is simple}}} N < M$

the unique largest semisimple submodule. $\quad$ $\underbrace{M/\mathrm{rad}(M)}_{\mathrm{top}(M)}$ is the unique largest semisimple quotient of $M$.

Apply to $\Lambda$ itself: $\mathrm{soc}({}_\Lambda\Lambda)$ and $\mathrm{rad}({}_\Lambda\Lambda)$ are not just left ideals of $\Lambda$, but two-sided ideals. $=: \mathrm{rad}(\Lambda)$

Note: $\mathrm{rad}({}_\Lambda\Lambda) = \mathrm{rad}(\Lambda_\Lambda)$ but $\mathrm{soc}({}_\Lambda\Lambda) \neq \mathrm{soc}(\Lambda_\Lambda)$

(6) $\mathrm{rad}(\Lambda) = 0$ $\quad$ (7) $\mathrm{soc}(\Lambda_\Lambda) = \Lambda$

Thm: $\mathrm{rad}(\Lambda)$ is the unique largest nilpotent ideal in $\Lambda$.

If an ideal of $J$ is nilpotent and $\Lambda/J$ is semisimple, $J = \mathrm{rad}\,\Lambda$

Note: $\mathrm{rad}\,\Lambda = \bigcap_{M\text{-simple}} \mathrm{ann}(M)$. Simples of $\Lambda \longleftrightarrow$ simples of $\Lambda/\mathrm{rad}\,\Lambda$ - finite!

Ex: $\Lambda = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in k \right\}$ $\quad$ $J := \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$ ideal

$\Lambda/J \cong k \times k$ semisimple. $J^2 = 0$ so $J = \mathrm{rad}(\Lambda)$.

$\mathrm{soc}({}_\Lambda\Lambda) = \left\{ \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} \right\}$ $\quad$ $\mathrm{soc}(\Lambda_\Lambda) = \left\{ \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \right\}$

$S = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$ is a left ideal, $\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$ is a two-sided ideal,

$\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$ is a right ideal $\quad$ (If $L$ is the regular rep $\begin{bmatrix} * \\ * \end{bmatrix}$

$0 \to S \to L \to T \to 0$

Radical series of $M$ (a $\Lambda$-module): $M \supseteq \operatorname{rad} M \supseteq \operatorname{rad}^2 M \supseteq \dots \supseteq \operatorname{rad}^{\ell_1} M = 0$

Here, $\operatorname{rad}^n M = \operatorname{rad}(\operatorname{rad}^{n-1} M) = (\operatorname{rad} A)^n \cdot M$

Socle series of $M$: $0 \subseteq \operatorname{soc} M \subseteq \operatorname{soc}^2 M \subseteq \dots \subseteq \operatorname{soc}^{\ell_2} M = M$

where $\operatorname{soc}^n M / \operatorname{soc}^{n-1} M = \operatorname{soc}\left( M / \operatorname{soc}^{n-1} M \right)$

Thm: $\ell_1 = \ell_2 =:$ Loewy length.

These series are Loewy series: The successive quotients are semisimple, and the length is the shortest possible among semisimple filtration.

Compare with the composition series, and Jordan Hölder

Notation: After enumerating the simples $S_1, S_2, S_3$ etc. people write things like $M = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 2 & \end{smallmatrix}$.

This means $M$ has a Loewy series

$$0 \subseteq U \subseteq V \subseteq M$$

s.t. $U \cong S_2$, $V/U \cong S_2 \oplus S_3$ and $M/V \cong S_1$

In this situation $\operatorname{soc}(M) = U \cong S_2$ $\operatorname{rad}(M) = V$.

$\operatorname{top}(M) := M/\operatorname{rad} M \cong S_1$