Before group reps, let me mention what kind of fin-dim algebras are "farthest" from being semisimple. k alg-closed Λ - f.d. k-alg.

Recall: $\operatorname{rad}\Lambda$ is the unique smallest ideal in Λ s.t. $\Lambda/\operatorname{rad}\Lambda$ is semisimple.
In particular, Λ is semisimple $\Leftrightarrow \operatorname{rad}\Lambda = 0$.

Defn: Λ is local if $\Lambda/\operatorname{rad}\Lambda \cong k$.

Being local is a very special property for an algebra. Being the antithesis of a semisimple alg. makes it look like it's a "dark side of the force" kind of special only, but they do come up! Definitely in alg. geo but also in rep theory

Prop: Let Λ be any f.d. algebra, and $M \in \underbrace{\Lambda\text{-mod}}$.
The f.d. algebra $\operatorname{End}_\Lambda(M)$ is local $\Leftrightarrow$ M is indecomposable

means left module, small "m" in mod means finite-dimensional

Prop: Λ is semisimple and local $\Leftrightarrow \Lambda = k$.

Analogy: Semisimple vs. solvable Lie algebras.

Prop: TFAE:
(1) Λ is local (1') Λ^{op} is local
(2) Λ has a unique maximal left ideal.
(3) ——————— right ideal
(4) Noninvertible elements in Λ form an ideal.
(5) Λ has a unique simple module.

Why (5)? Remember simples of $\Lambda \longleftrightarrow$ simples of $\Lambda/\operatorname{rad}\Lambda = k$.
Alternatively, $\Lambda^{op} \cong \operatorname{End}_\Lambda({}_\Lambda\Lambda)$ is local, so ${}_\Lambda\Lambda$ is indecomposable.
Hence only one indec. proj, so only one simple.

$(R_Q^n \subseteq I \subseteq R_Q^2)$

Prop: Let Q be a finite quiver, and I an admissible ideal of kQ

(1) kQ/I is semisimple $\iff$ Q has no edges

(2) kQ/I is local $\iff$ Q has a single vertex

Matches the intuition I think. Local $\Leftrightarrow$ only one vertex to focus.

Finally groups. Let G - finite group.

Prop: (1) kG is semisimple $\iff$ char $k \nmid |G|$.

(2) kG is local $\iff$ $|G| = (\text{char } k)^n$

I'm deliberately not using the letter 'p' because both statements hold when char $k = 0$! Exercise: Convince yourself that 1 is a power of 0.

If kG is local, rad kG = the augmentation ideal $:= \langle g-1 : g \in G - \{1\} \rangle_k$

Thm (Brauer): simple kG-modules $\longleftrightarrow$ conj. classes $[g]$ such that char $k \nmid |g|$

Brauer characters: p-prime char $k = p$. $g \in G$ is called p-regular if $p \nmid |g|$. Idea: Try to "save" the character theory in characteristic 0 by restricting to p-regular elements.

Find a local PID (R, m) such that $R/m \cong k$ and the field of fractions F of R has char $F = 0$, and F has a primitive $|G|$-th root of unity.

Ex. $k = \mathbb{F}_2 \twoheadleftarrow R = \mathbb{Z}_2 = \left\{ \frac{a}{b} \in \mathbb{Q} : 2 \nmid b \right\} \hookrightarrow \mathbb{Q}$ Not an appropriate example actually

I tried to find a reference for how to do this when $k = \bar{k}$, everyone seems to cite a book of Serre, which is French. Alg. closed is actually overkill, but we at least want some roots of unity in k and F.

Let $a = \text{lcm}\{g \in G : p \nmid |g|\}$.

Then there is a bijection $\{a^{th} \text{ roots of unity in } F\} \leftrightarrow \{a^{th} \text{ roots of unity in } k\}$

$\hat{\lambda} \longleftrightarrow \lambda$

Let $\rho: G \to GL_k(V)$ be a repn of G over k.
Given p-regular $g \in G$, $\rho(g)$ is diagonalizable, because we may regard V as a $k\langle g\rangle$-repn and $k\langle g\rangle$ is semisimple.
The eigenvalues $\lambda_1, \dots, \lambda_n$ are a^{th} roots of unity

Define $\chi_U : \{p\text{-regular elements}\} \longrightarrow F$

$$g \longmapsto \hat{\lambda}_1 + \dots + \hat{\lambda}_n$$

Why not use λ's in k? Taking trace in positive characteristic is a bad idea.

Satisfies a lot of the properties of usual characters. row/column orthogonality is a little different.

e.g. We'd like to have $\chi_U(1) = \dim U$ could be divisible by p!

What fails? $\chi_U = \chi_V$ does not imply $U \cong V$.

$$\chi_U = \chi_V \iff U \text{ and } V \text{ have the same multiset of composition factors}$$

inner product on p-regulars — pairs indec. proj with simples

Note: Brauer character theory is useless in the local case $|G| = p^a$.
Only $1 \in G$ is p-regular!

Application: Let U be a simple $\mathbb{C}G$-module, with character χ_U, s.t. $|G|_p$ divides $\dim U = \chi_U(1)$. Then for every $g \in G$ whose order is divisible by p (i.e. $g \in G - \{p\text{-regular elts}\}$) $\chi_U(g) = 0$.

S_3:

	1	(12)	(123)
1	1	1	1
sgn	1	-1	1
V	2	0	-1