The Okounkov - Vershik Approach to the Rep Theory of S_n

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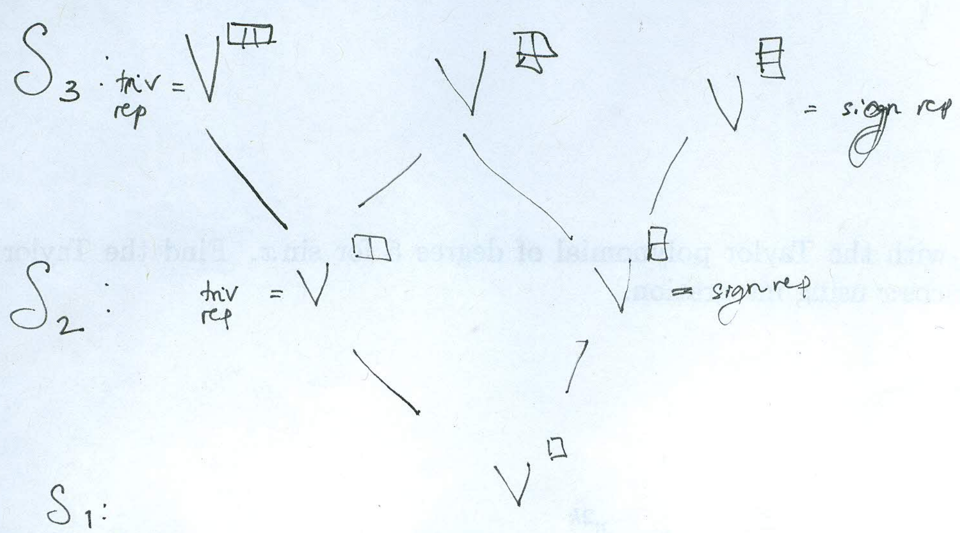
Summer Rep Theory Seminar.

Ex:



→ Ignoring what we know about tableaux

paths: $T_{(3)} = (\square \nearrow (2) \nearrow (3))$

$T_{(1,1,1)} = (\square \nearrow (1,1) \nearrow (1,1,1))$

$T_{(2,1)_1} = (\square \nearrow (2) \nearrow (2,1))$

$T_{(2,1)_2} = (\square \nearrow (1,1) \nearrow (2,1))$

$V_{T_{(3)}} = P_{\square} P_{(2)} P_{(3)} V^{(3)} \to V_{T_{(3)}}$ basis vector

$V_{T_{(1,1,1)}}$ basis vector of this 1-dim space

$V_{T_{(1,1,1)}} = P_{\square} P_{(1,1)} P_{(1,1,1)} V^{(1,1,1)}$

$V_{T_{(2,1)_1}} = P_{\square} P_{(2)} P_{(2,1)} V^{(2,1)}$

$V_{T_{(2,1)_2}} = P_{\square} P_{(1,1)} P_{(2,1)} V^{(2,1)}$

} can explicitly write basis vectors for these 1-dim'l spaces

$V^{(2,1)} = \left(\mathbb{C}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\right)^{\perp} \subset \mathbb{C}^3 \to$ defining representation

(S_3 acts by permuting coordinates)

To find $v_{T_{(2,1)_1}}$: action $S_2 \hookrightarrow S_3$ must be trivial

$\begin{bmatrix}1\\1\\-2\end{bmatrix} = V_{T_{(2,1)_1}}$

To find $V_{T_{(2,1)_2}}$: action $S_2 \hookrightarrow S_3$ sign rep

$\begin{bmatrix}1\\-1\\0\end{bmatrix} = V_{T_{(2,1)_2}}$

Later: action of YJM elements on these ~~basis~~ GZ-basis vectors.

Review & Spec(n)

1) RECALL: $GZ(n) := \langle Z(1), Z(2), \dots, Z(n) \rangle$

$$Z(i) := Z(\mathbb{C}S_i)$$ specifying to S_i now.

$T = \lambda_1 \nearrow \lambda_2 \nearrow \cdots \nearrow \lambda_n = \lambda$

$\lambda_i \vdash i$

a path on the branching graph

$V_T = P_{\lambda_1} \cdots P_{\lambda_n} V^{\lambda}$

$i \geq 2$ $\quad X_i \in GZ(n)$ $\quad (X_1 = id)$

$X_i = (1\,i) + (2\,i) + \cdots + (i-1\ i)$

Prop: $GZ(n) = \langle X_2, \dots, X_n \rangle$

As Theo mentioned, this algebra is important because it is the algebra of all operators diagonal in the GZ-basis

→ a GZ(-basis) vector is completely determined by the eigenvalues of elements of $GZ(n)$ on the vector.

→ So, to understand S_n representations we just need to understand the eigenvalues of $X_1, X_2, \dots, X_n$ acting on GZ-vectors.

→ NOTE: whole GZ-basis is a simultaneous eigenbasis for X_i's acting on $\mathbb{C}[S_n]$.

Ex: $V_{T_{\boxplus_1}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \in$ GZ-basis for $V^{\boxplus}$.

$X_1 = id$ $\qquad X_1 \cdot V_{T_{\boxplus_1}} = V_{T_{\boxplus_1}}$

$X_2 = (12)$ $\qquad X_2 \cdot V_{T_{\boxplus_1}} = V_{T_{\boxplus_1}}$ ⇝ projection onto trivial S_2 rep

$X_3 = (13) + (23)$ $\qquad X_3 \cdot V_{T_{\boxplus_1}} = (13)\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + (23)\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$$= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = -1 \cdot V_{T_{\boxplus_1}}$$

eigenvalues of X_1, X_2, X_3 acting on $V_{T_{\boxplus_1}}$

$(1, 1, -1)$

③

DEF: $v \in GZ$-basis for some irred-rep of S_n,

$$\alpha(v) := (a_1, a_2, \dots, a_n)$$

where $X_i \cdot v = a_i v$.

$$\mathrm{Spec}(n) := \left\{ \alpha(v) \;\middle|\; \begin{array}{l} v \text{ a GZ-basis vector} \\ \text{for some } S_n \text{ irred.} \end{array} \right\}$$

Equivalence relation on Spec(n)

$\alpha, \alpha' \in \mathrm{Spec}(n)$.

$\alpha \sim \alpha' \Leftrightarrow v_\alpha$ & $v_{\alpha'}$ are basis elements for the same irred-rep

Since the GZ-vectors are completely defined by the Spec. vectors, we often say given $\alpha \in \mathrm{Spec}(n)$, v_α is the corresp. GZ-vector

GOALS: 1.) Show that the Spec vectors are the content vectors for SYT, thus justifying the combinatorial interpretation.

2.) Construct matrices for the ~~reps~~ irred reps of S_n without tabloids, polytabloids, SYT, etc. only the action of the X_i's on the GZ-basis.

2.) ~~Local Action of the X_i's~~ Constructing Representations

The X_i's act locally on GZ-vectors:

Key part: The X_i's act locally on GZ-vectors:

PROP: v_T a GZ-vector, $T = \lambda_0 \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_n$)

$\lambda_i \mapsto i$

$1 \le k \le n-1$,

$$s_k \cdot v_T = \underbrace{\sum_{T' = \lambda_0 \nearrow \cdots \nearrow \lambda_{k-1} \nearrow \lambda_k' \nearrow \cdots \nearrow \lambda_n}}_{\text{paths that agree with } T \text{ except at the } k\text{-th level}} c_{T'} v_{T'}, \quad c_T \in \mathbb{C}$$

i.e. the action of s_k only affects the k^{th} level of the path

PROOF: $i > k$: $s_k \in S_i$ & $\mathbb{C}[S_i] v_T$ is irred.

$$\Rightarrow \underbrace{(\mathbb{C}[S_i]\, s_k)}_{\mathbb{C}[S_i] s_k = \mathbb{C}[S_i]} v_T = \mathbb{C}[S_i] \cdot v_T = V^{\lambda_i}$$

Theo's talk: b/c we project

$i<k \Rightarrow s_k$ commutes w/ every element in $\mathbb{C}[S_i]$

$$(\mathbb{C}[S_i]\, s_k) V_T = s_k V^{\lambda_i}$$
$$= \underbrace{V^{\lambda_i}}_{}$$

s_k acts S_i-linearly since it commutes with all elements of S_i $\square$

REM: In a similar way, the coefficients c_T, depend only on $\lambda_{k-1}, \lambda_k, \lambda_k', \lambda_{k+1}$

COR: i.) $s_i X_j = X_j s_i$, $j \neq i, i+1$
ii.) $s_i X_i s_i + s_i = X_{i+1}$

PROP. (Obvious & but helpful): $\mathbb{C}S_n = \langle \mathbb{C}S_{n-1}, X_{n-1}, X_n, s_n \rangle$.
w/ relations (i.), (ii.)

Proof: $\mathbb{C}S_n = \langle \mathbb{C}S_{n-1}, s_n \rangle$ $\square$

[The reason we add the superfluous generators is that it allows us to work inductively via the actions of the X_i's, which are what define the GZ-vectors]

So: Look at the actions of X_i, X_{i+1}, s_i in the space spanned by $\{V_T, s_i V_T\}$.

NOTE/REM: $\langle X_i, X_{i+1}, s_i \rangle \cong H(2)$, "degenerate affine Hecke algebra"

CASE 1: ~~$s_i \cdot V_\alpha = \pm V_\alpha$~~

$$X_i \longmapsto \begin{pmatrix} a_i & -1 \\ 0 & a_{i+1} \end{pmatrix}$$

$$X_{i+1} \longmapsto \begin{pmatrix} a_{i+1} & 1 \\ 0 & a_i \end{pmatrix}$$

$$s_i \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

from relations

diagonalize

(Assuming $s_i V_\alpha \neq \pm V_\alpha$,)

$$\left.\begin{aligned} X_i &\longmapsto \begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix} \\ X_{i+1} &\longmapsto \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix} \end{aligned}\right\} \text{diagonalized}$$

$$\underbrace{s_i \longmapsto \begin{pmatrix} \frac{1}{a_{i+1}-a_i} & 1-\frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i-a_{i+1}} \end{pmatrix}}_{\text{look familiar?}}$$

(*)

From these actions, we get the following properties.

<u>Prop</u>: $\alpha = (a_1, \dots, a_n) \in \mathcal{S}pec(n)$. Then $a_i \in \mathbb{Z}$ &

i.) $a_i \neq a_{i+1} \quad \forall i$

ii.) $a_{i+1} = a_i \pm 1 \Rightarrow s_i V_\alpha = \pm V_\alpha$

iii.) $a_{i+1} \neq a_i \pm 1 \Rightarrow s_i \alpha \in \mathcal{S}pec(n)$, $\alpha \sim s_i \alpha$,

$$V_{s_i \alpha} = \left(s_i - \frac{1}{a_{i+1} - a_i} \right) V_\alpha$$

& $(*)$ is the action of X_i, X_{i+1}, s_i in the basis $\underbrace{\{ V_\alpha, V_{s_i \alpha} \}}_{\text{i.e. this basis diagonalizes}}$.

<u>Note</u>: if $a_{i+1} \neq a_i \pm 1$, then in the basis

$$\underbrace{\Big\{ V_\alpha, \underbrace{(a_{i+1} - a_i)^{-1}}_{\overset{!!}{c_i}} \left(s_i - (1 - c_i^2)^{-1/2} I_d \right) V_\alpha \Big\}}_{\text{normalizes}}$$

the matrix of s_i is orthogonal & letting $r = a_{i+1} - a_i$,

$$s_i = \begin{pmatrix} \frac{1}{r} & \sqrt{1 - 1/r^2} \\ \sqrt{1 - 1/r^2} & -1/r \end{pmatrix} \Bigg\} \text{Young's Orthogonal Form}$$

<u>Young</u>: $r =$ "$\underbrace{\text{axial distance}}_{\text{difference of } \underline{\text{contents}}}$".

3.) Content Vectors & Tableaux

→ We have a way to explicitly construct the ired matrix representations of S_n & describe the action of the X_i's on the GZ-basis.

But: how do we actually know what these a_i's are?

Idea: Define content vectors, show they are exactly Spec(n) & both satisfy the same equivalence; content vectors $\longleftrightarrow$ SYT

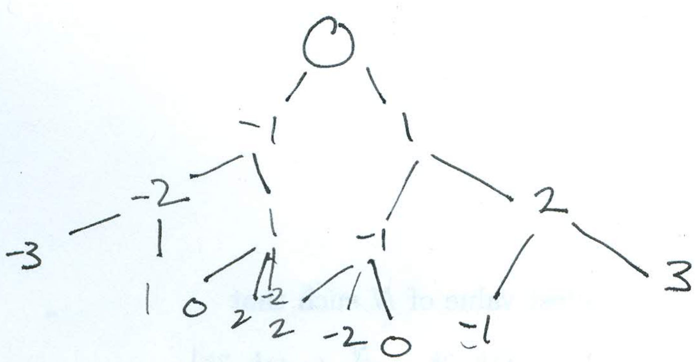
Def: $\alpha = (a_1, \dots, a_n) \in Cont(n)$ is a content vector if

i.) $a_1 = 0$

ii.) $a_m > 0 \Rightarrow a_i = a_m - 1$, some $i < m$
$a_m < 0 \Rightarrow a_i = a_m + 1$, some $i < m$

iii.) If $a_m = a_\ell = a$ for some $m < \ell$, then there must be $m \le i, j \le \ell$ such that $a_i = a-1$ & $a_j = a+1$

Ex: $(0, 1, -1, \boxed{-2}, 0, 2, 1, 3, -3)$



look familiar!

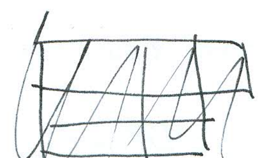
Proving $^{\text{Prop}}$ $\mathrm{Spec}(n) = \mathrm{Cont}(n)$ is basically a case-by case process. that use the relations of on the a_i.

Other Def: $\alpha, \beta \in \mathrm{Cont}(n)$, $\alpha \approx \beta \iff \beta = s_{i_1} s_{i_2} \cdots s_{i_k} \alpha$

→ eventually show the relations of Spec & Cont are the same.

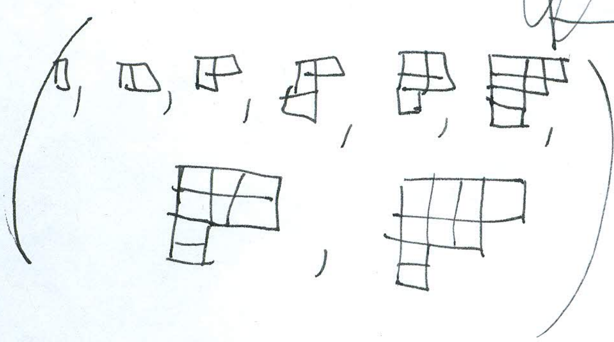
~~Prop~~

Def: SYT.



$\longleftrightarrow$

1	2	6	8
3	5	7	
4			
9			



Def: $C(\square) = x\text{-coord} - y\text{ coord}$. $T = (\nu_1, \nu_2, \nu_3, \ldots, \nu_n)$

$$\Updownarrow$$

$$C(T) = \left(c(\nu_2/\nu_1), c(\nu_3/\nu_2), \ldots, c(\nu_n/\nu_{n-1})\right)$$

0	1	2	3
-1	0	1	
-2			
-3			

Prop: Let $T = (\nu_0, \nu_1, \nu_2, \ldots, \nu_n)$ an SYT $\in SYT(n)$. The mapping

$$T \longmapsto C(T)$$

is a bijection of the set of $SYT(n)$ & $\mathrm{Cont}(n)$.

i.e. Every vector $v \in \mathrm{Cont}(n)$ defines a unique tableaux T, & for every $T \in SYT(n)$, $C(T) \in \mathrm{Cont}(n)$.

PROOF ~~[illegible]~~: just check conditions.

Ex: i.) $\underbrace{a_i \neq a_{i+1}.}_{\text{suppose}}$

$C(T) = (a_1, \dots, a_i, a_{i+1}, \dots, a_n)$ a content vector for T.

$a_i = a_{i+1}$

$\Rightarrow$ at $i+1^{st}$ step of path, add a box on the diagonal ✗

0	1	2	3
-1	0	1	
-2	-1	0	1

Punchline:
- We now know exactly how to calculate $\alpha(\pi_T)$ for T a path
- The branching graph of S_n is Young's lattice

M-N: Uses X_i's & actions