06/21/2016 Theo Douvropoulos Okounkov: Part 1

# Representation Theory of the Symmetric Group
## The Okounkov-Vershik approach.

0) Tradition is unsatisfactory!

Past approaches in constructing the irreducible Specht modules $V^\lambda$:

Think: Permutation action on letters of "banana"

1) i) Define permutation modules $M^\lambda$ for each partition $\lambda$. As $\lambda$ decreases in dominance order, each new $M^\lambda$ contains precisely one new irred. mod. $V^\lambda$ (that appears w/ multiplicity one.)

ii) $V^\lambda$ is the unique common component (simple mod) of $\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} id$ and $\uparrow_{\mathfrak{S}_{\lambda^T}}^{\mathfrak{S}_n} sgn$ → sign rep.

Young subgrp $\cong \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$

→ transpose of $\lambda$

(Thanks Vic, you were right!!)

google: David Speyer + "Construction of the representations of $S_n$" for a reference.

2) Duality between $\mathfrak{S}_n$ and $GL(k)$ in the space

$$\underbrace{\mathbb{C}^k \otimes \cdots \otimes \mathbb{C}^k}_{n \text{ times}}$$

Problems: a) These don't take into account:

• ) natural chain $\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots \subset \mathfrak{S}_n$

• ) $\mathfrak{S}_n$ is a coxeter group

Many ways to do that, but you need to fix one to even be able to talk about RSK, Young diagram, branching rule, etc...

b) Branching rule for restriction $\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$ and combinatorics of Young tableaux appear ad hoc: only after an obscure construction.

↳ definitely an OK phd-thesis

↳ Okounkov & Vershik's

Our approach does all of the above.

Ingredients:

- GZ-basis, a natural basis for irred. modules for any chain of grps where restriction is multiplicity-free.
- GZ-algebra, a/the maximal commutative subalgebra of $\mathbb{C}[G_n]$, an analog of the Cartan subalgebra from Lie theory
- An (abstract) purely algebraic proof that $\downarrow^{G_n}_{G_{n-1}} V^\lambda$ is multiplicity free.
- Young-Jucy-Murphy elements; algebra generators of the GZ-algebra that are diagonal w.r.t. the GZ-basis. Their spectrum behaves precisely like Young tableaux.

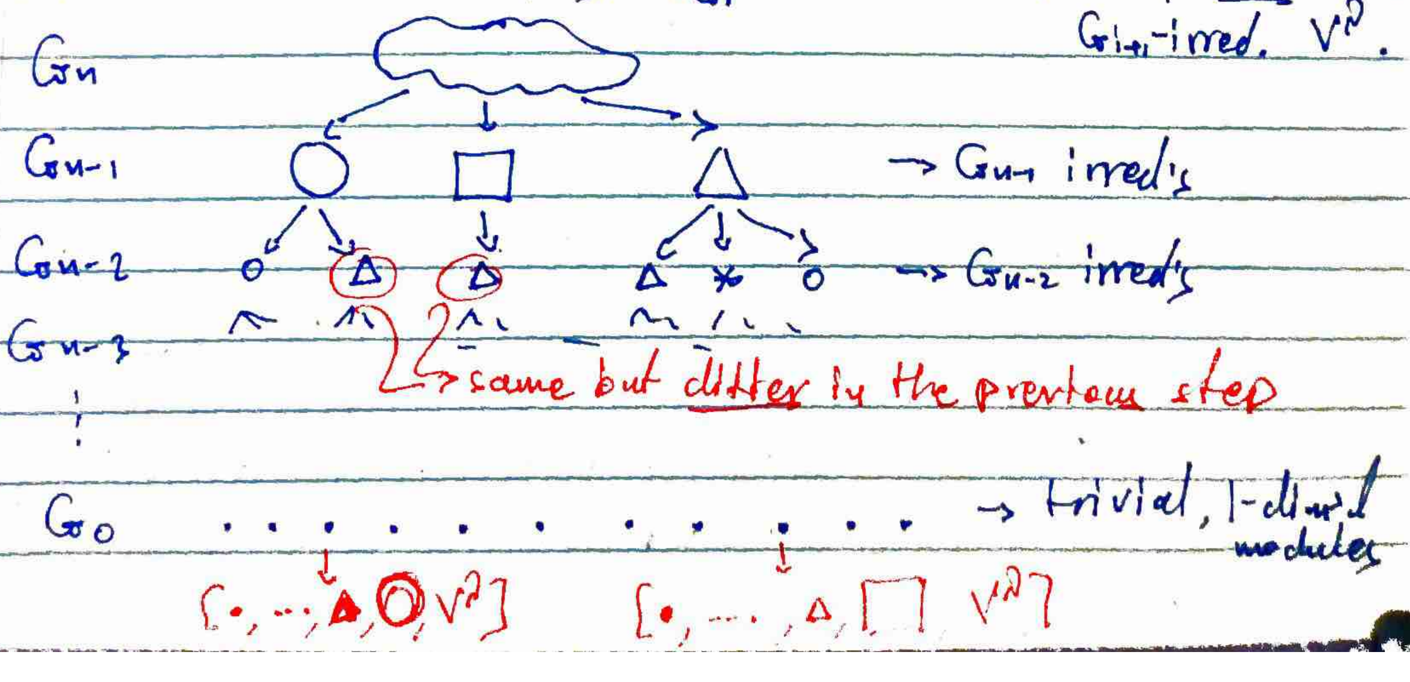
Think: Analogies ⟷ with weight spaces...

Applications:

- other chains of grps: $GL_n(\mathbb{F}_q)$, types B-C-D
- natural proofs for Murnaghan-Nakayama rule
- similar rules for other types?

applications anywhere we use rep. theory to do combin.

1) Gelfand-Tsetlin basis: Assume that a chain of grps $\{1\} = G_0 \subset G_1 \subset \dots \subset G_n$ satisfies $\downarrow^{G_{i+1}}_{G_i} V^\lambda$ is mult. free for any $G_{i+1}$-irred. $V^\lambda$.



Because Branching is simple, the decomposition

$$V^\lambda = \bigoplus_T V_T \; ; \quad T = \underbrace{\lambda^0 \uparrow \lambda^1 \uparrow \cdots \uparrow \lambda^i \uparrow \cdots \uparrow \lambda^n}$$

is canonical (each $V_T$ is an 1-dim'l space)

each $\lambda^i$ is a partition of $i$, and indexes a Specht module $V^{\lambda^i}$ of $S_i$ . . .

Choose a (unit if you may) vector $v_T$ in each 1-dim'l line $V_T$ and define GZ-basis $:= \{v_T\}$ where $T$ is a path in the branching graph (which we don't know yet will be the Young graph for the $S_n$-case)

Instead of paths $T$, we can use elements of the $\overbrace{\text{the}}^{\text{center of the}}$ grp algebra $\mathbb{C}[G_n]$ for indexing:

[Recall, if $V$ is any $G$-module ($G$ is finite), then the $\lambda$-isotypic component of $V$ is $P_\lambda \cdot V$ where $P_\lambda$ is a central ~~elt~~ idempotent of $\mathbb{C}[G_n]$.]

So, if $T = (\lambda^0 \uparrow \lambda^1 \uparrow \cdots \uparrow \lambda^n)$, $\lambda^n = \lambda$, the element $P_{\lambda^1} \cdot P_{\lambda^2} \cdots P_{\lambda} \in \mathbb{C}[G_n]$ acts on $V^\lambda$ by projecting on the line $V_T$. Call $P_T := P_{\lambda^1} \cdots P_{\lambda^n}$ .

Defn: The GZ-algebra $GZ(n) \subset \mathbb{C}[G_n]$ is given by $\langle Z(1), Z(2), .., Z(n) \rangle$ where $Z(i) = Z[\mathbb{C}[G_i]]$
i.e. it is algebraically generated by the centers of the grp algebras of the $G_i$

Now each $P_{j_i} \in Z[\mathbb{C}[G_i]]$ ($P_{j_i}$ is a central idempotent)

so $P_T = P_{j_1} \cdots P_{j_n} \in \langle Z(1), \ldots, Z(n) \rangle = GZ(n)$ for <u>all</u> paths $T$.

<u>Prop:</u> $GZ(n)$ is a/the maximal comm. subalgebra of $\mathbb{C}[G_n]$.

<u>Proof:</u>

- $GZ(n)$ is commutative: Indeed because it is generated by centers... → any elt $z_i$ commutes w/ any ~~polynom~~ monomial in the $z_j$'s $\left(z_1 z_n \cdot z_i \underset{\text{bc of } z_n}{=} z_1 \cdot z_i \cdot z_n \underset{\text{bc of } z_i}{=} z_i \cdot z_1 z_n\right)$ hence w/ any polynomial in $z_j$'s hence any two polynomials $p_1(\underline{z})$, $p_2(\underline{z})$ commute.
- Each $P_T$ is a projection on $V_T$.
  So the collection $\{P_T\} \subset GZ(n)$ is diagonal w.r.t. GZ-basis. Actually $GZ(n)$ contains <u>all</u> diagonal matrices w.r.t. GZ-basis (since $\exists\, P_T \ \forall T \ldots$).
  So $GZ(n)$ is a comm. subalgebra of $\mathbb{C}[G_n]$ that contains all diagonal matrices w.r.t. some basis; hence it is <u>maximal</u>.
  In particular it only contains diagonal matrices w.r.t. GZ-basis; in other words, all elements of $GZ(n)$ are simultaneously diagonalizable w.r.t. GZ-basis.

Remark: Any vector $v_T$ is uniquely determined by the eigenvalues of (all) the elts of $GZ(n)$ on it.
(i.e. $v_T$ is unique s.th. $P_T(v_T) = 1 \cdot v_T$, $P_{T'}(v_T) = 0 \cdot v_T$)
⊛ We don't need all the $P_T$'s; only some algebra-generators of $GZ(n)$

The following are specific to the $\mathfrak{S}_n$ case!

2) Jucy-Murphy elts:

$X_1 = id$

$X_2 = (12)$

$X_3 = (13) + (23)$

...

$X_i = (1i) + (2i) + \cdots + (i-1\ i)$

We mention the following without (... all proofs are easy) proof:

centre of $\mathbb{C}[\mathfrak{S}_n]$

Prop 1: $Z(n) \subset \langle Z(n-1), X_n \rangle$

⇝ combinatorically construct elts in $Z(n)$ for all conjugacy classes step by step. First single cycles, then all

Prop 2: $GZ(n) = \langle X_1, X_2, \ldots, X_n \rangle$

GZ-algebra for $\mathfrak{S}_n$ ...  ⇝ induction & prop 1.

Prop 3: $Z(\mathbb{C}[\mathfrak{S}_n], \mathbb{C}[\mathfrak{S}_{n-1}]) = \langle Z(n-1), X_n \rangle$

centralizer of $S_{n-1}$ in $S_n$ (grp algebras)

⇝ proof similar to Prop 1.

Corollary: $Z(\mathbb{C}[\mathfrak{S}_n], \mathbb{C}[\mathfrak{S}_{n-1}])$ is commutative!

Sidenote: (Jucys): The center $Z(\mathbb{C}[\mathfrak{S}_n])$ is generated by the symmetric polynomials in the elts $X_k$.

→ We are not using this; the proof might be difficult.

Remark: We'll see later, Prop 3, corollary is enough to guarantee multiplicity-free branching rule for $\mathfrak{S}_n$

Having chosen those very nice algebra-generators of $GZ(n)$, we now have a better way to index/identify the elts of the GZ-basis:

For each $v_T$, we consider the n-tuple $\alpha(v_T) = (a_1, \dots, a_n)$ s.th. $X_i \cdot v_T = a_i \cdot v_T$. Then call

$$\mathrm{Spec}(n) := \{\alpha(v_T) : v_T \text{ is a GZ-vector}\}$$

This set indexes the GZ-vectors much more efficiently and we can use it to describe all the $\mathfrak{S}_n$-irreducibles... More ~~details~~ on that by Elise...

3) Abstract-algebraic proof of the branching rule.

$\rightsquigarrow$ Why is $\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$ multiplicity free?

Proposition: Let $M$ be a semisimple, fin. dim'l $\mathbb{C}$-alg., $N$ a semisimple <u>subalgebra</u> $(N \leq M)$, and consider the centralizer $Z(M,N) := \{A \in M : AB = BA \ \forall B \in N\}$.

Then TFAE:

- $\downarrow^M_N V$ is mult. free for all $M$-irred's $V$.
- $Z(M,N)$ is commutative

In fact we know more: If $M \overset{\text{Artin-Wedderburn}}{\cong} \bigoplus_{\lambda \in \hat{M}} \mathrm{End}(W_\lambda)$, $N \cong \bigoplus_{\rho \in \hat{N}} \mathrm{End}(V_\rho)$ and the structure constants $m_{\lambda,\rho}$:

$$\downarrow^M_N W_\lambda = \bigoplus_{\rho \in \hat{N}} m_{\lambda,\rho} V_\rho \ , \text{ then}$$

An $m_{\lambda,\rho}$-by-$m_{\lambda,\rho}$ matrix...

$$Z(M,N) \simeq \bigoplus_{\lambda \in \hat{M}} \bigoplus_{\rho \in \hat{N}} M_{m_{\lambda,\rho}}(\mathbb{C})$$

For the "TFAE" part

Proof) · write $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$ (↑ matrix algebras)

· Let $N_i := \mathrm{Proj}(N)$ on $M_i$. notice $N_i$ is semisimple, since $N$ is...

Easily, $Z(M,N) = Z(M_1,N_1) \oplus \cdots \oplus Z(M_k,N_k)$

So, it is enough to show it for $Z(M_i,N_i)$, i.e. for the case $M$ is a matrix algebra (simple vs just semisimple...)

But now $M_i$ has only one irreducible, say $V_i$ and hence $M_i = \mathrm{End}(V_i)$, so $Z(M_i,N_i)$ is just the commutant algebra for the representation of $N_i$ given by the inclusion $N_i \le M_i$, which is precisely $\downarrow^{M_i}_{N_i} V_i$

Here we do know that $\downarrow^{\mathrm{End}(V_i)}_{N_i} V_i$ = mult. free $\iff$ $\mathrm{Comm}(N_i) = Z(M_i,N_i)$ = commutative ... ∎

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Q: Relation w/ Gelfand-Tsetlin polytopes...?

This comes from our theory but will be covered by Elise...

The dimension of the Specht module $V^\lambda$ is equal to the # of SYT of shape $\lambda$, which equals the # of integral pts in a polytope given by equations:

$$\begin{array}{cccccccc} x_{11} & < & x_{12} & < & x_{13} & < & x_{14} \\ \wedge & & \wedge & & \wedge & & \\ x_{21} & < & x_{22} & < & x_{23} & & \\ \wedge & & & & & & \\ x_{31} & & & & & & \end{array}$$

(and bounds $1 \le x_{ij} \le n$)

Similarly the # of integral pts of the Gelfand-Tsetlin polytope $GT(\lambda,\mu)$ equals the dimension of the weight-$\mu$ subspace of the irrep of $\mathfrak{gl}_n(\mathbb{C})$ w/ heighest weight $\lambda$.

$$GT(\lambda,\mu): \quad \begin{array}{ccccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & & & \lambda_n \\ & \rotatebox{90}{$\ge$}\ \rotatebox{90}{$\le$} & & \rotatebox{90}{$\ge$}\ \rotatebox{90}{$\le$} & & & & \rotatebox{90}{$\ge$}\ \rotatebox{90}{$\le$} & \\ & x_{1\,n-1} & & x_{2\,n-1} & & \cdots & & x_{n-1\,n-1} & \\ & & x_{1\,n-2} & & x_{2\,n-2} & \cdots & \xrightarrow{\text{sum}} & \mu_1+\mu_2+\cdots+\mu_{n} & \\ & & & & \vdots & & & & \\ & & & & x_{11} & \xrightarrow{\text{sum}} & \mu_1 & & \end{array}$$