$\mathbb{k} = \overline{\mathbb{k}}$ ①

A fin dim'l $\mathbb{k}$-alg. ~~Assume~~ $A = \mathbb{k}Q/I$ where (Q, I) is the bound quiver associated with A.

$\mathrm{rep}_{\mathbb{k}}(Q,I) \cong$

<u>Question</u>: How many indecomposable objects are in A mod?
What does the category A-mod look like?

If A mod has only finitely many, we say A is <u>representation-finite</u>

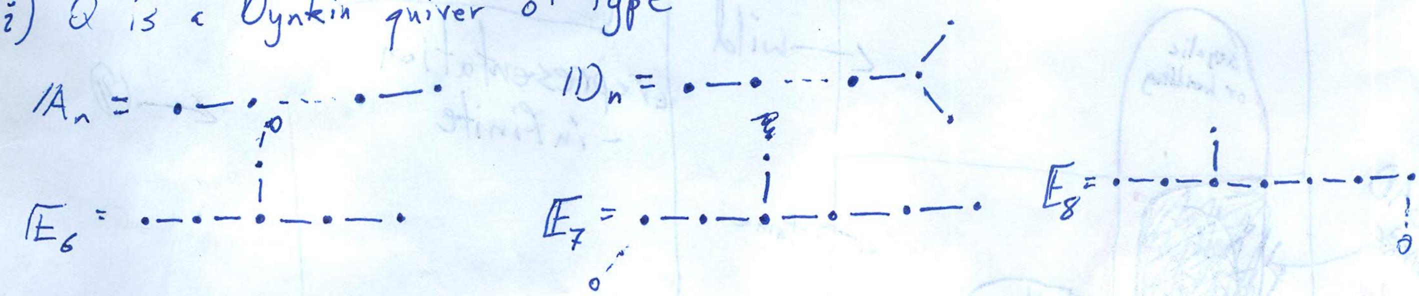
" " " infinitely " " <u>representation-infinite</u>.

(what can be said in the rep'n-finite case?)

<u>Thm</u> [Gabriel]: ~~Assume~~ Q is connected and $A = \mathbb{k}Q$. TFAE:

i) A is representation-finite

ii) Q is a Dynkin quiver of type



iii) $\cancel{q_A(\underline{x}) := \sum_{i \in Q_0} x_i^2 - \sum_{i \to j \in Q_1} x_i x_j + \sum_{(i,j) \in Q_0} x_i x_j \dim_{\mathbb{k}} \mathrm{Ext}^2(S_i, S_j)}$

$\sum_{i \in Q} m_i^2 - \sum_{i,j \in Q} \dim_{\mathbb{k}} \mathrm{Ext}^1_A(S_i, S_j)\, m_i m_j$ $n = \#(\text{simples of } A \text{ mod})$

$/\!/ \leftarrow$ for A hereditary

$\underline{\dim}(M) = (m_1, \dots, m_n) \in \mathbb{Z}^n$, $n = \#(\text{vertices of } Q)$

$$q_A(\underline{\dim}(M)) := \sum_{j=0}^{\infty} (-1)^j \dim_{\mathbb{k}} \mathrm{Ext}^j_A(M, M)$$

Euler form $q_A : K_0(A) \to \mathbb{Z}$. ← Grothendieck group

(defined when A has finite global dimension)

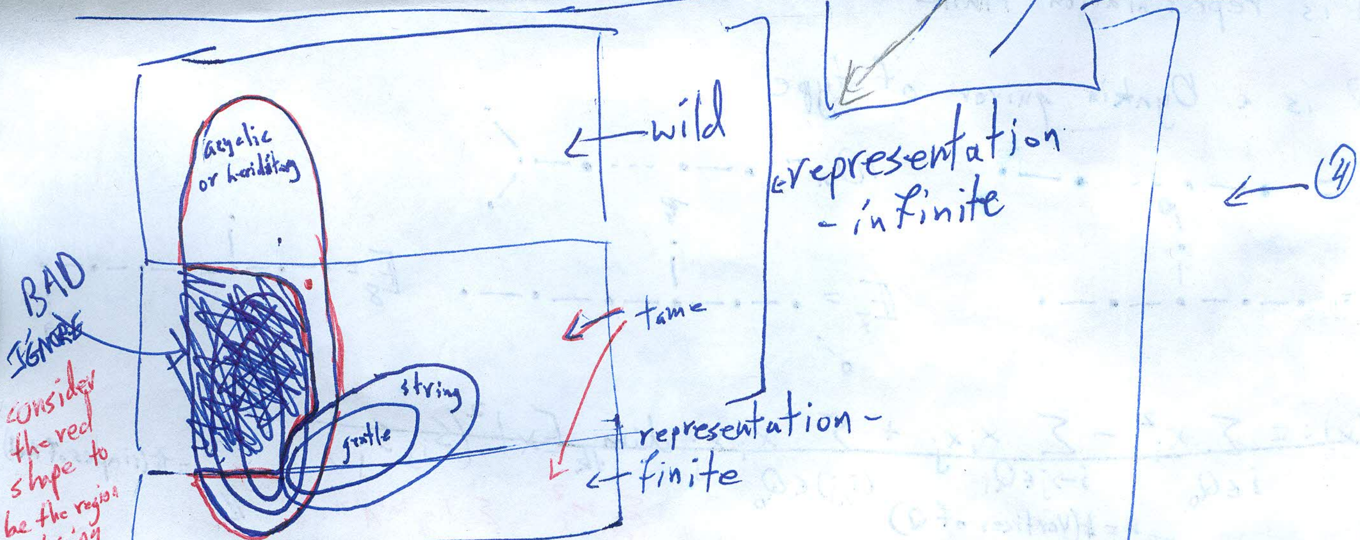
is positive definite (i.e. $q_A(\underline{x}) > 0$ for any $\underline{x} \neq \underline{0}$).

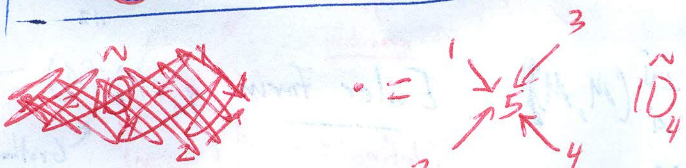
Furthermore, the <u>positive roots</u> (**) of q_A (i.e. $\underline{x} \in K_0(A)$ where $q_A(\underline{x}) = 1$) are in bijection with the indecomposable A-modules via $\underline{\dim}$

$V \longmapsto \underline{\dim}(V)$.

(**) The word positive root here is compatible with the corresponding notion for Dynkin diagrams in Lie theory.)

<u>Remark</u>: There is no general classification of indecomposable A-modules where A is representation finite.





For Dynkin quivers ~~In particular~~ each indecomposable is determined by its dimension vector. This enables one to use the <u>Knitting Algorithm</u> to compute the Auslander-Reiten quiver of A.

$$Q = 1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \qquad A = kQ$$

Projectives

$P_2 = 2 \qquad P_3 = {}^{3}_{2}$

$P_1 = {}^{1}_{\ 2} \qquad P_4 = {}_{2}{}^{3^{4}}$

Injectives

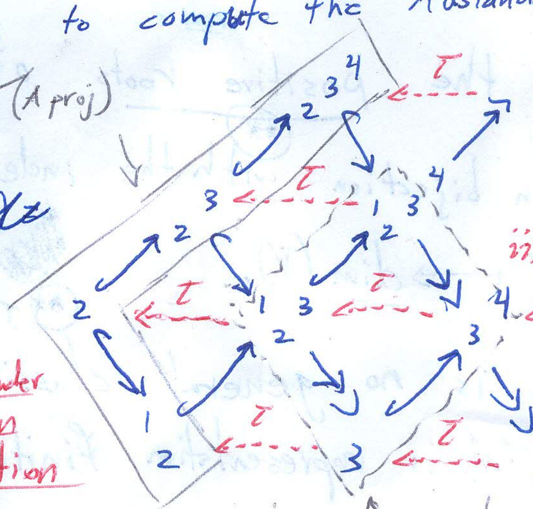
$I_1 = 1$

$I_2 = {}^{1}{}_{2}{}^{3^{4}}$

$I_3 = 3^{4}$

$I_4 = 4$

$\Gamma(A\,\mathrm{proj})$

$\Gamma(A\,\mathrm{mod}) =$ 

τ – Auslander Reiten translation

Fact:

i) $\tau X = 0 \iff X$ projective

ii) $\tau^{-1} X = 0 \iff X$ injective.

Outside of rep'n finite setting...

<u>Theorem:</u> (Drozd tame-wild dichotomy): A is either ~~tame or~~ of <u>tame</u> or <u>wild</u> representation type.

A is <u>tame</u> if for each $d \in \mathbb{N}$ there are at most finitely many one parameter families of d-dimensional indecomposable A-modules (with finitely many exceptions) (more precise def'n later)

A is <u>wild</u> if there exists an A-$\Bbbk[x]$-bimodule M s.t. M is a free $\Bbbk\langle x,y\rangle$ and the functor $M \otimes_{\Bbbk\langle x,y\rangle} - : \Bbbk\langle x,y\rangle \text{ mod} \longrightarrow A \text{ mod}$ sends distinct indecomposables to distinct indecomposables. (a <u>rep'n</u> <u>embedding</u>)

For any finite dimensional $\Bbbk$-algebra B there exists a fully faithful exact $\Bbbk$-linear functor

$$F : B \text{ mod} \longrightarrow \Bbbk\langle x,y\rangle \text{ mod}$$

exact: preserves exact sequences

$\Bbbk$-linear: $F_{X,Y} : \mathrm{Hom}_B(X,Y) \longrightarrow \mathrm{Hom}_{\Bbbk\langle x,y\rangle}(F(X),F(Y))$ is $\Bbbk$-linear; F preserves direct sums $F(X_1) \oplus F(X_2) \cong F(X_1 \oplus X_2)$

fully: $F_{X,Y}$ surjective

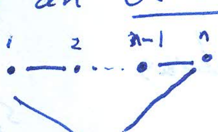
faithful: $F_{X,Y}$ injective

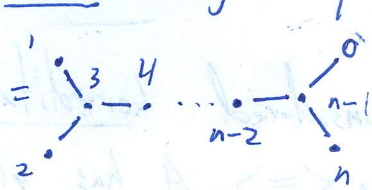
The composition $(M\otimes -)\circ F$ shows that the module category of B can be embedded into that of a wild algebra. (wild rep'n thy is essentially hopeless to understand)

<u>Theorem</u> [Kronecker, Nazarova, Donovan-Freislich]: Assume Q is connected, not Dynkin and $A = \Bbbk Q$. Then TFAE:

i) A is tame

ii) Q is an <u>extended</u> (or <u>affine</u>) Dynkin quiver of type

$\tilde{A}_n =$ 

$\tilde{D}_n =$ 

$\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

iii) q_A is positive semidefinite (i.e. $q_A(\underline{x}) \geq 0$ for all $\underline{x} \neq \underline{0}$)

8. If f is a scalar field and $\mathbf{F}$ is a vector field then which of the following expression is well defined?

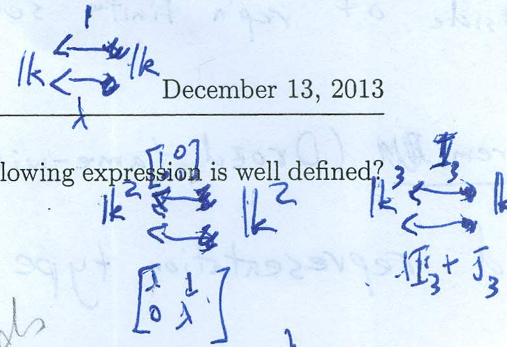
(a) div(f)

(b) div(curl f)

(c) div (div $\mathbf{F}$)

(d) curl (div $\mathbf{F}$)

(e) none of the above.



Kronecker quiver

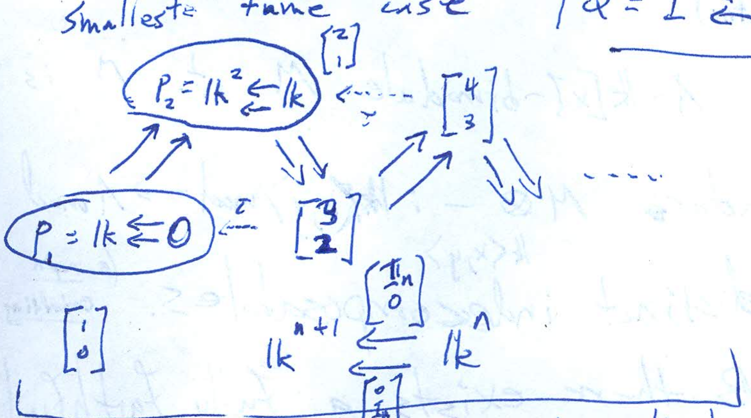
$Q = 1 \leftleftarrows 2$

Kronecker algebra.

$\Bbbk(1 \leftleftarrows 2)$

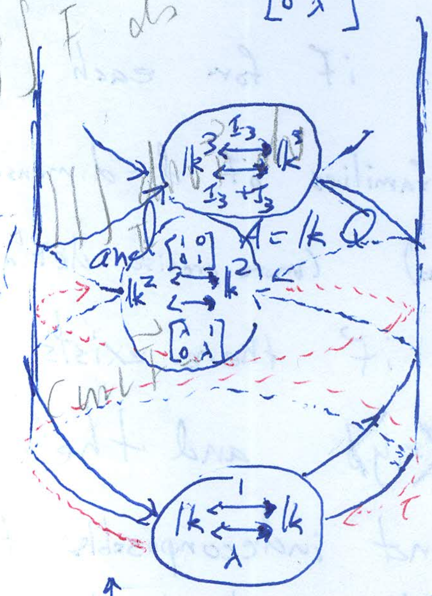
(rep'n infinite)

Smallest tame case



preprojective (or postprojective)

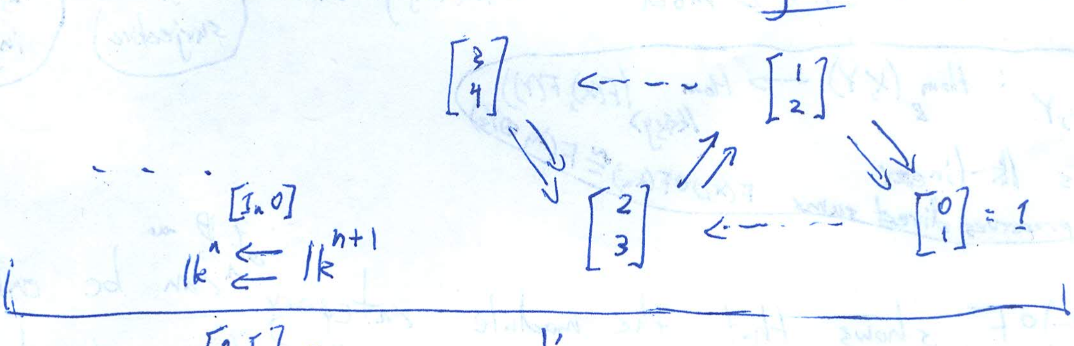
$\mathcal{P}$



one tube for each $(a:\lambda) \in \mathbb{P}^1/\Bbbk$

$\mathcal{R}$

regular component



$\mathcal{I}$ preinjective

finitely many one parameter families of indecomposables of a given dimension. (can get finitely many exceptions by considering $A = \Bbbk(\cdot \cdots \leftleftarrows \cdot)$)

$A\text{mod} = \mathcal{P} \cup \mathcal{R} \cup \mathcal{I}.$

$q_A(\underline{x}) = x_1^2 + x_2^2 - 2x_1x_2$

$q_A\left(\dim_{\Bbbk}\left(\Bbbk \overset{1}{\underset{1}{\leftleftarrows}} \Bbbk\right)\right) = 0$

"Minimal" examples of wild algebras are

$\Bbbk(\cdot \Lleftarrow \cdot)$

$\Bbbk(\cdot \leftleftarrows \cdot \leftarrow \cdot)$

We have so far only considered hereditary algebras $A = \Bbbk Q$ where Q acyclic and A finite dim'l $\iff$ A has gldim ≤ 1 $\iff$ All ~~subobjects~~ submodules of projectives are projective.