Quiver algebras

> **Throughout:** K is an algebraically closed field.
> Modules will be <u>left</u>-modules.

Let Q be a quiver (directed graph, loops & multiple edges are o.k.)

Def The <u>quiver algebra</u> KQ is the K-vector space with basis the set of paths of length ≥ 0 in Q.

$$\beta \cdot \alpha := \begin{cases} \beta\alpha & (\alpha \text{ followed by } \beta) \\ & \text{if these paths concatenate} \\ 0 & \text{o/w} \end{cases}$$

Remark This is the category algebra of the free category generated by Q.

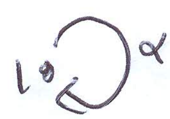
Examples

a) $$Q = \overset{1}{\bullet} \xleftarrow{\gamma} \overset{2}{\bullet} \xleftarrow{\beta} \overset{3}{\bullet} \xleftarrow{\alpha} \overset{4}{\bullet}$$

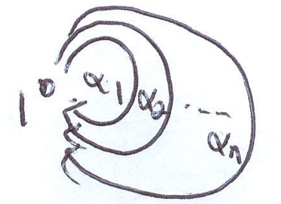
$$KQ = \operatorname{span}\{\alpha, \beta, \gamma, \beta\alpha, \gamma\beta, \gamma\beta\alpha, \underbrace{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4}_{\text{paths of length } 0}\}$$

$$(\gamma\beta + 4\gamma)\alpha = \gamma\beta\alpha + \cancel{4\gamma\alpha} = \gamma\beta\alpha.$$

Examples

b) $Q =$ 

$$kQ = \text{span}\{1, \alpha, \alpha^2, \dots\}$$
$$\cong k[t]$$

c) $Q =$ 

$$kQ = k\langle \underbrace{t_1, \dots, t_n}_{\text{noncommuting variables}} \rangle$$

Remark KQ is not always finite-dimensional, but there is a sufficient condition on two-sided ideals I such that KQ/I is finite-dimensional.

nhb

Let R_Q be the two-sided ideal spanned by all paths of length ≥ 1. Then

(R_Q^k has all paths of length $\geq k$).

(Prop) $R_Q^n \subseteq I \subseteq R_Q^2 \implies \dfrac{KQ}{I}$ is finite-dimensional

for some n.

In this case we say I is <u>admissible</u>. (It is convenient to restrict our attention to admissible ideals.)
(two-sided)

<u>Question</u> What do (finitely-generated) modules over KQ/I <u>look</u> like?

They are obtained as follows:

① Draw Q.

② Place a f.d K-vectorspace at each vertex of Q.

③ Define linear maps at each arrow which satisfy the relations in I.

<u>Ex</u> $Q = \overset{1}{\bullet} \xleftarrow{\gamma} \overset{2}{\bullet} \xleftarrow{\beta} \overset{3}{\bullet} \xleftarrow{\alpha} \overset{4}{\bullet}$ $\qquad A := KQ/\langle \gamma\beta\alpha \rangle$

Let $M = \underset{\operatorname{span}(x)}{\overset{\shortparallel}{K}} \xleftarrow{f_\gamma = [0\ 1]} \underset{\operatorname{span}(y,z)}{\overset{\shortparallel}{K^2}} \xleftarrow{f_\beta = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \underset{\operatorname{span}(u)}{\overset{\shortparallel}{K}} \xleftarrow{f_\alpha = \mathrm{Id.}} \underset{\operatorname{span}(v)}{\overset{\shortparallel}{K}}$. (Indeed, $f_\gamma f_\beta f_\alpha = 0$)

This is a 5-dimensional space, where for example γ acts as the 5x5 matrix

$$\begin{array}{c} x \\ y \\ z \\ u \\ v \end{array} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\quad\ \ x \quad y \quad z \quad u \quad v$$

The simple KQ/I-modules are

$$"1" = L(1) = S(1) \cong K \leftarrow 0 \leftarrow 0 \leftarrow 0$$

$$S(2) \cong 0 \leftarrow K \leftarrow 0 \leftarrow 0$$

$$S(3) \cong 0 \leftarrow 0 \leftarrow K \leftarrow 0$$

$$S(4) = 0 \leftarrow 0 \leftarrow 0 \leftarrow K$$

all maps zero.

So KQ/I is basic ($\dim S = 1 \ \forall S$)

The semisimple KQ/I-modules are of the form

$$K^{n_1} \xleftarrow{0} K^{n_2} \xleftarrow{0} K^{n_3} \xleftarrow{0} K^{n_4}.$$

(Place any vector space at each vertex, define all maps to be zero)

The regular representation (KQ/I as a left KQ/I-module, given by left multiplication).

Is given as follows.

1. At each vertex, place a copy of K for each path ending there
2. Define maps "according to left multiplication."

$\underline{\text{Ex}}$ $A = KQ/I$ where $Q:\ \overset{1}{\bullet} \xleftarrow{\gamma} \overset{2}{\bullet} \xleftarrow{\beta} \overset{3}{\bullet} \xleftarrow{\alpha} \overset{4}{\bullet}$, $I = \langle \gamma\beta\alpha \rangle$.

$$
{}_A A = K^3 \xleftarrow{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} K^3 \xleftarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}} K^2 \xleftarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} K
$$

$$
\begin{array}{cccc}
\| & \| & \| & \| \\
\operatorname{span}(\varepsilon_4, \gamma, \gamma\beta) & \operatorname{span}(\varepsilon_3, \beta, \beta\alpha) & \operatorname{span}(\varepsilon_2, \alpha) & \operatorname{span}(\varepsilon_1)
\end{array}
$$

$$
= \begin{array}{ccccccc}
 & \gamma & & \beta & & \alpha & \\
0 & \longleftarrow & K\beta\alpha & \longleftarrow & K\alpha & \longleftarrow & K\varepsilon_4 \\
\oplus & & \oplus & & \oplus & & \oplus \\
K\gamma\beta & \longleftarrow & K\beta & \longleftarrow & K\varepsilon_3 & \longleftarrow & 0 \\
\oplus & & \oplus & & \oplus & & \oplus \\
K\gamma & \longleftarrow & K\varepsilon_2 & \longleftarrow & 0 & \longleftarrow & 0 \\
\oplus & & \oplus & & \oplus & & \oplus \\
K\varepsilon_1 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}
$$

The direct summands are the <u>projective</u> <u>indecomposables</u>:

$$P(4) = 0 \longleftarrow K \xleftarrow{1} K \xleftarrow{1} K$$

$$P(3) = K \xleftarrow{1} K \xleftarrow{1} K \longleftarrow 0$$

$$P(2) = K \xleftarrow{1} K \longleftarrow 0 \longleftarrow 0$$

$$P(1) = K \longleftarrow 0 \longleftarrow 0 \longleftarrow 0$$

The <u>injective</u> indecomposables are the direct summands of $\overbrace{\mathrm{Hom}(\underbrace{A_A}_{\text{right } A\text{-module}}, K)}^{\text{left-module}}$.

$\mathrm{Hom}(A_A, K)$ is constructed as follows!

① At vertex i, place a copy of K for each path <u>beginning</u> there.

② Define maps according to "the dual of right multiplication."

In our example, $\mathrm{Hom}(A_A, K) =$

$$\left.\begin{array}{ccccccc}
0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & K\varepsilon_4 \\
 & & & & \oplus & & \\
0 & \leftarrow & 0 & \leftarrow & K\varepsilon_3 & \leftarrow & K\alpha \\
 & & & & \oplus & & \\
0 & \leftarrow & K\varepsilon_2 & \leftarrow & K\beta & \leftarrow & K\beta\alpha \\
 & & & & \oplus & & \\
K\varepsilon_1 & \leftarrow & K\gamma & \leftarrow & K\gamma\beta & \leftarrow & 0
\end{array}\right| \begin{array}{l} = I(4) \\ \oplus \\ = I(3) \\ \oplus \\ = I(2) \\ \oplus \\ I(1) \end{array}$$

Here, e.g $I(1) = \mathrm{span}([\varepsilon_1], [\gamma], [\gamma\beta])$, but e.g

$$\beta \cdot [\gamma\beta] = [\gamma]$$

(un-multiply by β since the path begins with β).

Yep, it's a left action.

The radical? $\quad \mathrm{rad}(kQ/I) = R_Q$

$$\mathrm{rad}^k(kQ/I) = R_Q^{\,k}$$

And, as before, $\mathrm{rad}^k(M) = \mathrm{rad}^k(kQ/I)M = R_Q^k M.$

$$\Downarrow$$

"vectors acted on by paths of length k"

Socle?

$\mathrm{soc}^k(M) =$ "vectors that are annihilated by every path of length k"

Note! $\quad \dim\left(\dfrac{R_Q}{R_Q^2}\right) =$ total # of edges in Q.

$\dim \varepsilon_i\left(\dfrac{R_Q}{R_Q^2}\right)\varepsilon_j =$ total # of edges from i to j.

<u>Main structure theorem:</u> Let A be a fd $\mathbb{K}$-algebra which is basic ($\dim S = 1$ $\forall$ S simple) and connected ($A \cong A_1 \times A_2$ only trivially).

Then there is a quiver Q such that $A \cong KQ/I$ for some admissible ideal I.

<u>Proof sketch</u>. Let $e_1, \ldots, e_n$ be a complete system of pw orthogonal idempotents.

Define Q as follows:

- Vertices $1, \ldots, n$ in bijection with idempotents.
- The number of edges from i to j is $\dim\left(e_i \frac{\mathrm{rad} A}{\mathrm{rad}^2 A} e_j\right)$

We get a quiver algebra KQ and a map $KQ \xrightarrow{\phi} A$ sending

$$\varepsilon_i \longmapsto e_i$$

$$(\text{each edge } \alpha) \longmapsto \left(\text{the corresponding basis vector of } \frac{\mathrm{rad} A}{\mathrm{rad}^2 A}\right).$$

This <u>is well-defined</u> and <u>surjective</u>!

And $\ker \phi$ is an admissible 2-sided ideal.

$\blacksquare$