Bourbaki, Lie..., ch4 §2 +ex. 22, 24

Mathas, Iwahori-Hecke algebras, Schur algebras, ...

I. Iwahori-Hecke Algebras in the wild

1. Convolutions

q - power of prime

$G = GL_n(\mathbb{F}_q)$

$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$

$\mathcal{F}(G) = \{f : G \to \mathbb{C}\}$

$$\mathcal{F}(G/B) = \{f : G \to \mathbb{C} \mid f(gb) = f(g)\} = \mathrm{triv}\uparrow_B^G$$

$$\mathcal{F}(B\backslash G/B) = \{f : G \to \mathbb{C} \mid f(b'gb) = f(g)\}$$

Convolution: $$(f_1 * f_2)(g) = \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

$$*: \mathcal{F}(G) \times \mathcal{F}(G/B) \longrightarrow \mathcal{F}(G/B)$$

$$\mathcal{F}(B\backslash G/B) \times \mathcal{F}(B\backslash G/B) \to \mathcal{F}(B\backslash G/B) \quad (\text{really } \mathcal{F}(B\backslash G) \times \mathcal{F}(G/B) \to \mathcal{F}(B\backslash G/B))$$

Ex:

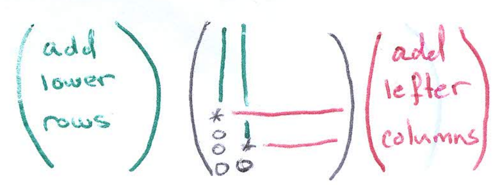
$$(f_1 * f_2)(b'gb) = \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(\underbrace{x^{-1}b'}_{y^{-1}}gb) = \frac{1}{|B|} \sum_{y \in G} f_1(b' \ y) f_2(y^{-1}gb) =$$

$$= \frac{1}{|B|} \sum_{y \in G} f_1(y) f_2(y^{-1}g) = (f_1 * f_2)(g)$$

$$\mathcal{H} := (\mathcal{F}(B\backslash G/B), *)$$

2. Presentation

Bruhat decomposition: $$G = \overset{\text{disjoint union}}{\coprod_{w \in S_n}} BwB$$



$[S_n = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1 \ \vdots \rangle$, $\ell(w) = $ min # adj. transpositions needed to write w.$]$

Basis of $\mathcal{H}$: $\{T_w\}_{w \in W}$, where $T_w(g) = \begin{cases} 1 & g \in BwB, \\ 0 & \text{else.} \end{cases}$

Claim: $T_s * T_s = (q-1)T_s + qT_1$

$T_s * T_w = T_{sw}$ if $\ell(sw) > \ell(w)$

Proof thoughts: $$T_s * T_s(g) = \frac{1}{|B|} \sum_{x \in G} T_s(x) T_s(x^{-1}g) = \frac{1}{|B|} \# \{x \in BsB \mid x^{-1}g \in BsB\}$$

$g \in BsBBsB$

Lemma 1: $BsBBsB = B \cup BsB$

Lemma 2: $|BwB| = q^{\ell(w)}|B|$

$\begin{pmatrix} \boxed{+} & * & * \\ & \boxed{+} & * \\ & & \boxed{+} \end{pmatrix}$ $\begin{pmatrix} * & \boxed{+} & * \\ \boxed{+} & * & * \\ & & \boxed{+} \end{pmatrix}$ only can't be one thing to be indep. of first col.

If $g \in B$, $(T_s * T_s)(g) = \dfrac{|BsB|}{|B|} = q$.

If $g \in BsB$, $(T_s * T_s)(g) = \dfrac{1}{|B|} \# \{x \in BsB \mid x^{-1}g \notin B\} = \dfrac{q|B| - |B|}{|B|} = q-1$

So $T_s * T_s = (q-1)T_s + qT_1$.

The claim reduces to a presentation

$$\mathcal{H} \cong \left\langle \{T_s\}_{s \in S} \;\middle|\; \begin{array}{l} T_s^2 = (q-1)T_s + qT_1 \\ T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}} \\ T_{s_i}T_{s_j} = T_{s_j}T_{s_i} \quad , |i-j| \geq 2 \end{array} \right\rangle$$

3. Endomorphisms

$$\mathcal{F}(G) \times \mathcal{F}(G/B) \xrightarrow{*} \mathcal{F}(G/B) \Rightarrow \mathcal{F}(G/B) \times \mathcal{F}(B \backslash G/B) \xrightarrow{*} \mathcal{F}(G/B) \Rightarrow$$

we have a map $\mathcal{F}(B\backslash G/B) \to \mathrm{End}_{\mathbb{C}}(\mathcal{F}(G/B))$.

Image is G-equivariant (recall $(g.f_1)(g') = f_1(g^{-1}g')$)

$$((g.f_1) * f_2)(g') = \frac{1}{|B|} \sum_{x \in G} f_1(\underbrace{g^{-1}x}_{y}) f_2(x^{-1}g') = \frac{1}{|B|} \sum_{y \in G} f_1(y) f_2(y^{-1}g^{-1}g) =$$

$$= (g.(f_1 * f_2))(g')$$

The map is a "anti" ring map (associativity of convolution).

Thm: $\mathcal{H} \cong (\mathrm{End}_G(\mathcal{F}(G/B)))^{op}$

II Cellularity

Def A-algebra $(\mathcal{H})$, Λ-poset (partitions w/ dominance), $\{T(\lambda)\}_{\lambda \in \Lambda}$ (standard Young tableaux)

Suppose $\{C^{\lambda}_{s,t}\}_{\substack{\lambda \in \Lambda \\ s,t \in T(\lambda)}} \subseteq A$ is a basis. It is cellular if

(1) $C^\lambda_{s,t} \to C^\lambda_{t,s}$ is an algebra automorphism,

(2) $\forall \lambda \in \Lambda,\ t \in T(\lambda),\ a \in A$

$$C^\lambda_{s,t}\, a \equiv \sum_{u \in T(\lambda)} r^a_{ut}\, C^\lambda_{su} \pmod{A^{>\lambda}}$$

r^a_{ut} does not depend on s!

$C^\lambda_s := \mathrm{span}\{c^\lambda_{st} + A^{>\lambda}\}_{t \in T(\lambda)}$ in $A^{\geq \lambda}/A^{>\lambda}$ does not $\overbrace{\text{depend}}^{\text{"really"}}$ on s.

Example: $\mathcal{H}_3$, $\Lambda = $ (3) — (2,1) — (1,1,1)

$s = \boxed{1|2|3}$, $t = \begin{smallmatrix}1&2\\3\end{smallmatrix}$, $u = \begin{smallmatrix}1&3\\2\end{smallmatrix}$, $v = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$

$C^{(3)}_{ss} = 1 + T_1 + T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1$

$C^{(2,1)}_{tt} = 1 + T_1$

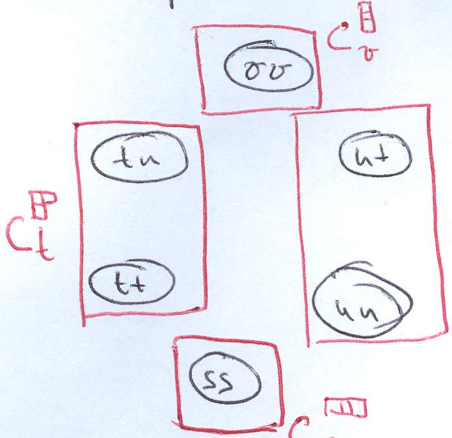
$C^{(2,1)}_{tu} = (1+T_1)T_2$

$C^{(2,1)}_{ut} = T_2(1+T_1)$

$C^{(2,1)}_{uu} = T_2(1+T_1)T_2$

$C_{vv} = 1$

Get a filtration on the basis:



$C^{(2,1)}_t A \leq C^{(2,1)}_t + \cancel{C^{(3)}_{\dots}}$ quotient out

$C_{tu} \cdot T_1 = -c_{tu} - C_{tt} + \cancel{c_{ss}}$

Thm (Graham & Lehrer '96). Can produce all irreducibles explicitly.

(there is a form $\langle , \rangle$ on C^λ, $C^\lambda/\ker\langle , \rangle$ is 0 or irrep, and these are all the irreps)