Remarks

original motivation ↙

- Murphy's basis of $\mathcal{H}$ is cellular; so is Lusztig's canonial basis (type A!)
- Murphy's basis lifts to basis of q-Schur algebra; get that it is quasi hereditary
- König, Xi ('98): Suppose A cellular. Then

  A is quasi-hereditary $\Leftarrow\!\Rightarrow$ A has fin. global. dim.

  $(\mathrm{Ext}^i_A(X,Y)=0$ for $i>>0)$

- Du, Rui ('98): Quasihereditary w/ certain type of involution $\Rightarrow$ cellular.

III W-graph representations of $\mathcal{H}$

1. W-graphs

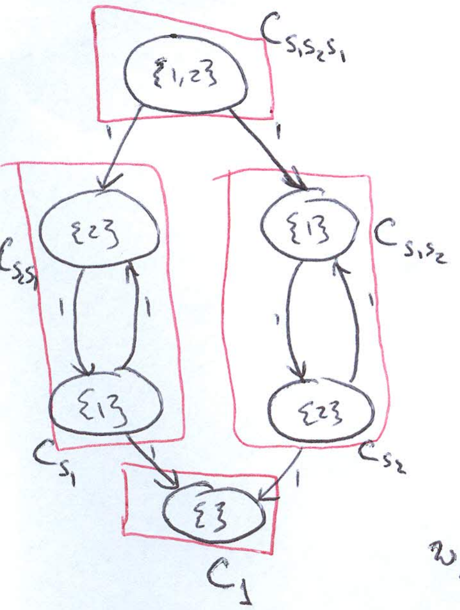
(W, S) - Coxeter system $\left[W = \langle s \in S \mid s^2 = 1,\ \underbrace{stst\cdots}_{m_{st}} = \underbrace{tsts\cdots}_{m_{st}} \rangle \right]$

Ground ring: $\mathbb{Z}[q^{\pm 1/2}]$

<u>Def</u>. A <u>W-graph</u> is a triple (X, m, τ) where $(X, m: X\times X\setminus \text{diagonal} \to \mathbb{Z})$ is a weighted directed graph w/out loops, $\tau: X \to 2^S$, such that span(X) carries an $\mathcal{H}$-rep. given by

$$T_s(\overset{\text{vertex}}{u}) = \begin{cases} -u & s\in\tau(u) \\ qu + q^{1/2}\sum\limits_{v:\, s\in\tau(v)} m(v\to u)\, v & ,\ s\notin\tau(u)\end{cases}$$

Example:



$$T_{s_1} C_{s_1s_2} = -C_{s_1s_2}$$

$$T_{s_2} C_{s_1s_2} = q\,C_{s_1s_2} + q^{1/2}\left(C_{s_2} + C_{s_1s_2s_1}\right)$$

Cells:= strongly connected components of W-graph.

These also define a filtration of basis w/ subquotient representations on cells.

Remarks

1. Why "C_w"? That is how $\mathcal{H}$ acts on Lusztig's canonical basis, a basis fixed by an involution called the bar involution $\left(\bar{q} = q^{-1}, \overline{T_w} = (T_{w^{-1}})^{-1}\right)$, and whose expansion in $\{T_w\}$ is unitriangular and relatively nice.
2. There is a "most important W-graph" for each W, it has vertices $\{C_w\}$, $\tau(C_w)$ = descent set of w, m is given by coefficients of Kazhdan–Lusztig polynomials. Goal: get your hands on this graph w/o computing KL polynomials.
3. Quadratic relation satisfied for all graphs; important relations are the braid ones.
4. In this example filtration is the opposite of the one we had in Part II. To fix this may take transpose of W-graph action.
5. In type A, "cellularity" and "KL cells" give the same filtration. This was the motivation for defining cellularity. Then cellularity was applied to other algebras: Schur algebras, some Brauer algebras, Cyclotomic Hecke algebras (Ariki–Koike), Temperley–Lieb algebras, BMW algebras, ...). But in other Lie types KL-cells do <u>not</u> give cellularity. Only in (Geck '07) was it shown that $\mathcal{H}$ is in general cellular.

6. Combinatorics: under mild assumption can try to classify W-cells; e.g. (Stembridge '12) showed that there are finitely many.

2. Gyoja's Theorem

Thm ('84) For any finite Coxeter group W, any irrep of $\mathcal{H}_W$ ($\mathbb{Q}\otimes$) has a W-graph basis.

Proof idea: define (≥ will do) another algebra $\mathcal{G}$ with $\mathcal{H} \subset \mathcal{G}$ such that $\mathcal{G}$-modules "are" (≥ will see) W-graphs. Show that the inclusion has a left (≥ will not do) inverse $\mathcal{G} \twoheadrightarrow \mathcal{H}$. Then every $\mathcal{H}$-module becomes a $\mathcal{G}$-module, i.e. a W-graph.

Def
$$\mathcal{G}' := \left\langle \begin{array}{c} e_s, x_s \\ (s \in S) \end{array} \;\middle|\; \begin{array}{l} e_s^2 = e_s \\ e_s e_t = e_t e_s \\ e_s x_s = x_s \\ x_s e_s = 0 \\ (x_s^2 = 0) \end{array} \right\rangle$$

"e_s = projection on vertices whose τ contains s"

"x_s = take all edges to given vertex over which s is lost"

Want $\mathcal{H} \to \mathcal{G}'$ s.t. $T_s \mapsto -e_s + q(1-e_s) + q^{1/2} x_s$. While quadratic relation is satisfied, braid relations are not. So take quotient:

$$\mathcal{H} \to \mathcal{G} := \mathcal{G}' / \text{?}.$$

Explicit generators of ?: Hahn '14.

In fact, $\mathcal{C}_y$ is a path algebra of a quiver on 2^S w/ $|I/J|$ arrows

$I \leftarrow J$; vertex elements are

$$E_I := \prod_{t\in I} e_t \prod_{t'\in S\setminus I} (1-e_{t'})$$

(projection on verts whose τ is I)

and edge elements

$$X_{IJ}^{s} = E_I\, x_s\, E_J \;,\quad s\in I\setminus J.$$

W-Graphs → $\mathcal{C}_y$-modules

(↓ vertex)

$$e_s . x = \begin{cases} x & s\in \tau(x) \\ 0 & s\notin \tau(x) \end{cases} \qquad x_s . x = \begin{cases} \sum_{y : s\in\tau(y)} y\, m(y\to x) & , s\notin \tau(x) \\ 0 & , s\in\tau(x) \end{cases}$$

$\mathcal{C}_y$-modules → W-graphs

For $I\subseteq S$, choose basis B_I of $E_I V$.

Vertices $:= \bigcup_{I\subseteq S} B_I$

$\tau(B_I) = I$

Read m off the action on Basis.