Representation Stability in the Cohomology of Configuration Spaces:

① Configuration spaces ③ A local to global thm

② FI-modules presented in finite degree

X - a topological space

$n \in \{0, 1, 2, \dots\}$

$\mathrm{Conf}_0(X) = \emptyset$

$\mathrm{Conf}_1(X) = X$

$$\underbrace{\mathrm{Conf}_n(X) = \{(x_1, \dots, x_n) : x_i \neq x_j\}}_{\text{subspace topology}} \subseteq \underbrace{X^n}_{\text{product topology}}$$

$[n] = \{1, 2, \dots, n\}$

Comes up in several areas in math.

Functorial in n. We can write $\mathrm{Conf}_n(X) = \underbrace{\mathrm{Inj}}_{\text{injective functions}}([n], X)$

For an injection $[m] \to [n]$, we get a natural map

$$\mathrm{Inj}([n], X) \longrightarrow \mathrm{Inj}([m], X) \quad \text{"forgets points"}$$

Yields a contravariant functor from $\underbrace{FI}_{\text{finite sets and inj.}}$ to Top

$\mathrm{Conf}_\bullet(X)$

Fix $i \in \{0, 1, \dots\}$. $H^i(-; \mathbb{Z})$: contravariant from Top to $\mathbb{Z}$-Mod,

we get a covariant functor

$$H^i(\mathrm{Conf}_\bullet(X)) : FI \longrightarrow \mathbb{Z}\text{-Mod}.$$

an FI-module!

Or, $H^*(\mathrm{Conf}_\bullet(X)) : FI \longrightarrow$ Graded rings $\sim$ a graded FI-algebra

Thm (Fred Cohen, 70s): For $d \geq 2$, the FI-algebra $H^*(\mathrm{Conf}_\bullet(\mathbb{R}^d))$ is generated (as an algebra) by a single element in degree 2!

If $V : FI \to \mathbb{Z}\text{-Mod}$ is finitely generated, each $V_n := V([n])$ has to be fin. gen. as an abelian group.

Defn (incomplete): An FI-module V is presented in finite degree if

① it is generated in finite degree, that is, it has a generating set $S \subseteq \bigsqcup_{n=0}^{N} V_n$ for some N.

② the "relations" of V also lives in finite degree.

Equivalently, there exist "free" FI-modules P, Q generated in finite degree and an exact sequence means induced from FB

$$P \to Q \to V \to 0$$

Remark: A more flexible condition than finite generation. Though less descrip for the individual S_n-reps.

Ex: If $M \in \text{FI-Mod}$ is finitely generated, $\bigoplus M$ is infinite no longer fin-gen. but is still finitely presented.

Thm (Church-Ellenberg) FI-modules presented in finite degree are closed under taking submodules, quotients, extensions.

(implies Noetherianity of FI-mod)

Thm (B.): Suppose X is a space with an open covering $\mathcal{U}$ such that

① $\mathcal{U}$ is basis like: Given $U, V \in \mathcal{U}$ and $x \in U \cap V$, $\exists W$ s.t. $x \in W \subseteq U \cap V$.

② Every finite subset $S \subseteq X$ is contained in some $U \in \mathcal{U}$.

③ $\mathcal{U}$ only has finitely many homeomorphism types.

④ For every $U \in \mathcal{U}$, and $i = 0, 1, \ldots$ $H^i(\mathrm{Conf}_\bullet(U))$ is presented in finite degree

Then for every $j = 0, 1, \ldots$ $H^j(\mathrm{Conf}_\bullet(X))$ is presented in finite degree

Proof: Set up a spectral sequence of FI-modules and propagate finite presentation using [CE]

Remark: X = a connected d-dimensional manifold with $d \geq 2$
$\mathcal{U}$ = Euclidean neighborhoods
satisfy the conditions: (1), (3) evident. (2) elementary
(4) Cohen $\longrightarrow$ As a corollary,

Thm (Church et al): For every manifold M and $j \geq 0$
$H^j(\mathrm{Conf}_\bullet(M))$ is presented in finite degree.

What to do? Establish stability for other spaces. Seems harder than I first hoped.
Improve stability ranges using this new approach: