On Attempts to Categorify the Cluster Structure

Reiten - Cluster Category - sections 1-3

Outline

1. Intro
2. Attempt 1 - BGP Functor
3. Attempt 2 - APR - Tilting
4. Attempt 3 - Orbit ~~cat~~ of Bounded Derived Category

1. Intro - What is a Quiver? What is a cluster algebra?

Quiver: directed graph, no loops, 2-cycles for this talk

Cluster algebra: Given underlying quiver n-vertices, have n variables

Each has a mutation property - will illustrate with an example

$\{x_1, x_2, x_3\}$ $\quad 1 \to 2 \to 3$

$$\mu_i(x_i) = x_i' = \frac{\prod_{j\to i} x_j + \prod_{i \to j} x_j}{x_i}$$

$\mu_i(Q)$ - flip arrows incident to Q

- for $j \to i \to k$ add $j \to k$
- delete 2-cycles

$\xrightarrow{\mu_2}$ $\left\{x_1, \frac{x_1+x_3}{x_2}, x_3\right\}$ $\quad 1 \leftarrow 2 \leftarrow 3$, $3 \to 1$ $\xrightarrow{\mu_1}$ $\left\{\frac{\left(\frac{x_1+x_3}{x_2}\right)+x_3}{x_1}, \frac{x_1+x_3}{x_2}, x_3\right\}$ $\quad 1 \to 2 \; \cancel{\leftarrow} \; 3$, $3 \to 1$

$\xrightarrow{\mu_3}$ $\left\{x_1, x_2, \frac{x_2+1}{x_3}\right\}$ $\quad 1 \to 2 \leftarrow 3$

The cluster algebra is the algebra generated by all the cluster variables

<u>Goal</u>: Develop a category (objects w/ morphisms) that can emulate the "cluster structure", i.e., have a mutation property like quivers & clusters.

(quivers will appear as endomorphism algebras)

Attempt #1

A representation of a quiver involves assigning vector spaces to vertices and linear maps to the edges.

e.g. $K \xrightarrow{1} K \xrightarrow{\binom{1}{0}} K^2$

~~Bernstein-Gelfand-Ponomarev introduced a~~

One example of a category is rep Q, where objects are representations and morphisms are an n-tuple of linear maps ~~between X, Y~~ between corresponding vector spaces.

$$\begin{tikzcd} V_1 \arrow[r, "f_\alpha"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\#"] & V_2 \arrow[r, "f_\beta"] \arrow[d, "h_2"] \arrow[dr, phantom, "\#"] & V_3 \arrow[d, "h_3"] \\ W_1 \arrow[r, "g_\alpha"] & W_2 \arrow[r, "g_\beta"] & W_3 \end{tikzcd}$$

Moreover, Bernstein-Gelfond-Ponomarev introduced a functor from rep Q to rep Q', where Q' is $\mu_i(Q)$ (i must be a sink!)

Example: $V_1 \xrightarrow{f_\alpha} V_2 \xrightarrow{f_\beta} V_3 \longrightarrow V_1 \xrightarrow{f_\alpha} V_2 \xhookleftarrow{i} \ker f_\beta$

(The BGP functor was used in a proof of Gabriel's Theorem

Thm [Gabriel]
A quiver Q is of finite mutation type iff the underlying graph is type ADE.)

Thm [BGP] The reflection functor F_i induces an equivalence F_i: rep Q/S_i to rep Q'/S_i', $Q' = \mu_i(Q)$

Illustrate w/ AR-Quiver.

Thm [Auslander-Reiten] For any indecomp, non proj Λ-module C; (Λ f.d. k-algebra)
$\exists!$ a "almost split" sequence $0 \to A \to B \to C \to 0$. Then, we denote $\tau(C) = A$ the AR-translation & similarly $\tau^{-1}(A) = C$.
Split is $0 \to A \to A \oplus C \to C \to 0$. Almost split is technical...

Using the AR-translation, for a quiver Q, we can draw a nice picture of the indecomposable representations & irreducible morphisms of rep Q.

$$\text{rep } Q \rightarrow \begin{tikzcd}[column sep=small] & P(2) \arrow[dr] \arrow[r] & P(1)=I(3) \arrow[dr] & \\ S(3)=P(3) \arrow[ur] & & S(2) \arrow[r] & I(2) \arrow[r] & I(1) \end{tikzcd}$$

$$\xleftarrow{\ \tau\ }$$

$$\text{rep } Q' \quad \begin{tikzcd}[column sep=small] & P(1) \arrow[dr] & & I(3)=S(3) \\ P(2)=S(2) \arrow[ur] \arrow[r] & P(3) \arrow[r] & I(2) \arrow[ur] \arrow[r] & I(1)=S(1) \end{tikzcd}$$

See an equivalence between categories w/ appropriate modding out.

Examples of almost split sequences: $P(3) \to P(2) \to S(2)$

$P(2) \to S(2) \oplus P(1) \to I(2)$

?? "$P(2)$ is an extension of $P(3), S(2)$"

Good: can do some mutations in category

Bad: cannot mutate at arbitrary vertices * (Dw2 ...)

no clear analog of cluster vs cluster variables.

Attempt #2

rep Q is equivalent to mod kQ

Moreover, thinking of kQ as a module over itself, can write $kQ = P(1) \oplus \dots \oplus P(n)$ (includes each path once)

And, $\mathrm{End}_{kQ}(kQ) \simeq Q^{op}$ (Retain information about quiver)

Thm [APR] (P=Platzeck)

a) For $T = kQ/P_i \oplus \tau^{-}(P_i)$, we have $\mathrm{End}_{kQ}(T)^{op} \simeq kQ'$

(not as important) b) The functor F_i: rep $Q \to$ rep Q' is isomorphic to the functor $\mathrm{Hom}_{kQ}(T, -)$: mod$_{kQ} \to$ mod kQ'

Illustrate End quiver flip @ AR quiver.

$$\begin{tikzcd} P_1 \oplus P(2) \oplus P(3) = kQ \arrow[r, mapsto] \arrow[d] & T = P(1) \oplus P(2) \oplus \tau^{-1} P(3) \arrow[d] \\ Q \arrow[r, mapsto, "\mu_3"] & \mu_3(Q) = Q' \end{tikzcd}$$

Good news: Have much clearer analogs of cluster, cluster variables, and quiver, & corresponding mutations.

Note kQ, T are examples of tilting modules

To be tilting, module over kQ: $\underbrace{\mathrm{Ext}^1_{kQ}(T,T)=0}_{\text{rigid}}$ & # indecomposable summands $= n = \#$ vertices

Mutation is rarely as easy as τ^{-1}.

Bad news: In mod kQ cannot always find T_i^* so that $T/T_i \oplus T_i^*$ is tilting, and our quiver information gets lost.

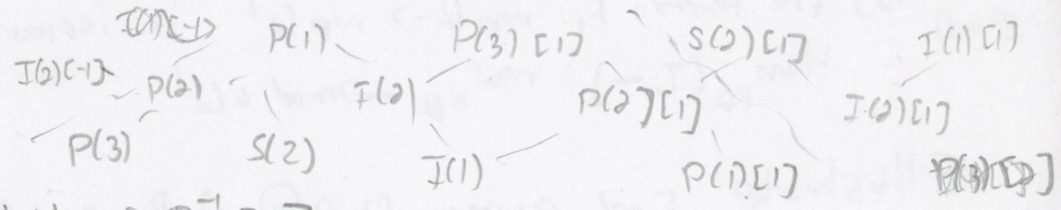
For example, to mutate @ $P(2)$, choose $I(1)$. But

$\mu_2(Q) = 1 \leftarrow 2 \leftarrow 3$ & $\mathrm{End}_{kQ}(P(1)\oplus I(1)\oplus P(3))^{op} = \leftarrow 2 \quad 3$

Even worse, there is no replacement for $P(1)$.
We are lacking indecomposable objects & irreducible morphisms.

4. Orbit Category of Bounded Derived Category

Introduce shift functor, $X \mapsto X[1]$, to get a much larger category



Consider the orbit of $\tau^{-1}[1]$

$\tau^{-1}[1](P(3)) \to S(2)[1]$

" $(P(2)) \to I(2)[1]$

$(P(1)) \to P(3)[2]$

In Bounded Derived category $D_b(kQ)$

ind Objects $X[i]$ X ind in mod kQ, $i \in \mathbb{Z}$

morphisms $X[i] \to Y[i]$ if $X \to Y$ & between injectives $[i-1]$ proj $[i]$

In orbit category $= \mathcal{C}_Q$

ind objects: equivalence classes under $F = \tau^{-1}[1]$ (fundamental domain)

morphisms: $\operatorname{Hom}_{\mathcal{C}_Q}(X, Y) = \bigoplus_{i=-\infty}^{\infty} \operatorname{Hom}_{D_b(kQ)}(X, F^i Y)$

(Good news, this is 0 unless $i = 0, 1$)

~~Back~~ More good news - # ind in $\mathcal{C}_Q$ = # cluster variables.

Back to previous examples:

~~now, End~~ want to recalculate $\operatorname{Hom}_{\mathcal{C}_Q}(I(1), P(3))$

$$= \operatorname{Hom}(I(1), P(3)) \oplus \operatorname{Hom}(I(1), \tau^{-1}[1]P(3))$$
$$= 0 \oplus \operatorname{Hom}(I(1), S(2)[1])$$
$$\simeq k$$

And now we can use $\mu_1(P_1 \oplus P_2 \oplus P_3) = P(1)[1] \oplus P(2) \oplus P(3)$

Thm [BMRRT]

The cluster category $\mathcal{C}_Q$ has cluster structure w.r.t. the cluster tilting objects

(maximal rigid)

Thm [R]

Let Q be acyclic finite quiver, Q' mut equiv to Q. Let $i \in Q'_0$.

Then $\exists$ cluster tilting object T' in $\mathcal{C}_Q$ s.t. for $T'' = \mu_i(T')$,

$$\begin{tikzcd}
T' \arrow[r, "\mu_i"] \arrow[d, dash, "\mathrm{End}^{op}"'] & T'' \arrow[d, dash, "\mathrm{End}^{op}"] \\
Q_{T'} = Q' \arrow[r, mapsto, "\mu_i"] & \mu_i(Q') = Q_{T''}
\end{tikzcd}$$