S-Griffeth 10/24/06

For $G(2,1,1) = \{1, s\}$

$\mathbb{H}_c$ has gens x, y, t_s and relations

$$t_s^2 = 1, \quad t_s x t_s^{-1} = -x, \quad t_s y t_s^{-1} = -y$$

Choose $\alpha_s = x$, $\alpha_s^\vee = 2y$, $\langle x, y\rangle = 1$

$$yx - xy = K - c_s \langle x, y\rangle\langle x, 2y\rangle t_s$$
$$= K - 2c_s t_s$$

$M_c(\mathbb{1})$ has basis $\{1, x, x^2, \dots\}$

w/ action
$$x \cdot x^n = x^{n+1}$$
$$t_s \cdot x^n = (-1)^n x^n$$
$$y \cdot x^n = Knx^{n-1} - c_s \langle \alpha_s, y\rangle \frac{x^n - sx^n}{\alpha_s}$$
$$= Knx^{n-1} - c_s \frac{x^n - (-1)^n x^n}{x}$$
$$= \begin{cases} Knx^{n-1} & n \text{ even} \\ (Kn - 2c_s)x^{n-1} & n \text{ odd} \end{cases}$$

CLAIM: If $Kn - 2c_s \neq 0 \ \forall n \in 2\mathbb{Z}_{\geq 0} + 1$, $M_c(\mathbb{1})$ is irred.

proof: Later ∎

Suppose for some odd positive n, $Kn - 2c_s = 0$

Then $x^n, x^{n+1}, \dots$ span an $\mathbb{H}_c$-submodule

since $y \cdot x^n = (Kn - 2c_s)x^{n-1} = 0$

Claim: $L_c(\mathbb{1})$ ($:=$ simple head of $M_c(\mathbb{1})$)

$$= \mathbb{C}[x]/(x^n)$$

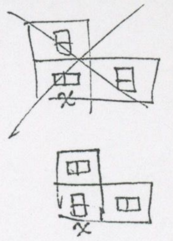
i.e. $\mathbb{C}\{x^n, x^{n+1}, \dots\}$ is the unique maximal submodule of $M_c(\mathbb{1})$

Claim: Any $\mathbb{H}_c$-submodule of $M_c(\mathbb{1})$ is graded

proof:
$$xy \cdot x^n = \begin{cases} Knx^n & \text{if } n \text{ even} \\ (Kn - 2c_s)x^n & \text{if } n \text{ odd} \end{cases} \quad \text{for generic } K, c_s$$

$$\Rightarrow \big(\underbrace{xy + c_s(1 - t_s)}_{h}\big) \cdot x^n = \begin{cases} Knx^n & \text{if } n \text{ even} \\ Knx^n & \text{if } n \text{ odd} \end{cases}$$

So $M = h\text{-eigenspaces} = \mathbb{C}\{x^i\}$ ∎



Take $\kappa = 1$. $\quad \overbrace{(2m+1)}^{n} - 2c_s = 0 \qquad m \in \mathbb{N}_{\geq 0}$

i.e. $c_s = m + \frac{1}{2}$

So $\mathbb{C}[x]/(x^{2m+1})$ is an irred. $\mathbb{H}_c$-module.

<u>$m=1$</u>: $\kappa = 1$, $c_s = \frac{3}{2}$

$L_c(\mathbb{1})$ has basis $1, x, x^2$

w/ action

$$\begin{array}{llll} & t_s : 1 \mapsto 1 & x : 1 \mapsto x & y : 1 \mapsto 0 \\ & x \mapsto -x & x \mapsto x^2 & x \mapsto -2 \\ & x^2 \mapsto x^2 & x^2 \mapsto 0 & x^2 \mapsto 2x \end{array}$$

x spans the unique copy of the sign rep'n in $L_c(\mathbb{1})$.

Since $L_c(\mathbb{1})$ is irreducible,

$$L_c(\mathbb{1}) = \mathbb{H}_c . x$$

Recall $\mathbb{H}_c$ has relations

$$t_w x t_w^{-1} = wx, \quad t_w y t_w^{-1} = wy \quad t_v t_w = t_{vw}$$

$$xy - yx = \kappa \langle x, y \rangle - \sum_{s \in T} c_s \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle t_s$$

Setting $\deg(x) = \deg(y) = 1$ $\deg t_w = 0$ $\forall x \in \mathfrak{h}^*, y \in \mathfrak{h}, w \in W$

gives a <u>filtration</u> $\mathbb{H}_c^{(n)} = \mathrm{span}\{f(x) t_w g(y) : \deg f + \deg g \leq n\}$

i.e. $\mathbb{H}_c^{(n)} \mathbb{H}_c^{(m)} \subset \mathbb{H}_c^{(n+m)}$

[Setting $\deg(x) = 1$, $\deg(y) = -1$, $\deg(t_w) = 0$ gives a <u>grading</u>]

We get a filtration on $L_c(\mathbb{1})$...

$$L_c(\mathbb{1})^{(0)} = \mathbb{H}_c^{(0)} . x = \mathbb{C}x$$

$$L_c(\mathbb{1})^{(1)} = \mathbb{H}_c^{(1)} . x = \mathbb{C}1 \oplus \mathbb{C}x \oplus \mathbb{C}x^2$$

$\mathrm{gr}\, \mathbb{H}_c$ acts on $\mathrm{gr}\, L_c(\mathbb{1})$

$\quad \parallel$

$S(\mathfrak{h} \oplus \mathfrak{h}^*) \otimes_{\mathbb{C}} \mathbb{C}W$

→ it is usual that an algebra is a deformation of its assoc. graded ⇝ tangent cone....

$$\mathrm{gr}\{L_c(\mathbb{1})\} = \quad \begin{tikzcd}[column sep=small] & 0 & & 0 & & 0 & & 0 & \\ \text{deg } 1: & & \mathbb{C}1 \arrow[ul, "y"] \arrow[ur, "x"'] & & & & \mathbb{C}x^2 \arrow[ul, "y"] \arrow[ur, "x"'] & & = \text{trivial } \mathbb{C}W\text{-mod} \\ \text{deg } 0: & & & & \mathbb{C}x \arrow[ull, "y=(-2)"] \arrow[urr, "x"'] & & & & = \text{sign } \mathbb{C}W\text{-mod} \end{tikzcd}$$

So $S(\mathfrak{h}\oplus\mathfrak{h}^*)$ acts with $x^2,\ xy,\ y^2$ annihilating
$\quad\ \ \shortparallel$
$\mathbb{C}[x,y]$

so
$$L_c(\mathbb{1}) = S(\mathfrak{h}\oplus\mathfrak{h}^*)\otimes_{\mathbb{C}}W\cdot x$$
$$= \underbrace{S(\mathfrak{h}\oplus\mathfrak{h}^*)}_{\mathbb{C}[x,y]}\cdot x$$

is a quotient of $\mathbb{C}[x,y]/(x^2,xy,y^2)$,
$S(\mathfrak{h}\oplus\mathfrak{h}^*)/(S(\mathfrak{h}\oplus\mathfrak{h}^*)^W_+)$

For general positive m,

$$\begin{tikzcd}[row sep=small] 1 \arrow[r, "x"] & 0 & x^{2m} \arrow[l, "y"'] & & \underline{W\text{-rep'n}} \\ & & & & (\mathrm{sign})^0 \\ x \arrow[uu, "y"] \arrow[r, "x"] & 0 & x^{2m-1} \arrow[l, "y"'] \arrow[uu] & & (\mathrm{sign})^1 \\ \vdots \arrow[u, "y"] & & \vdots \arrow[u, "x"'] & & \vdots \\ x^{m-1} \arrow[u, "y"] \arrow[r, "x"] & 0 & x^{m+1} \arrow[l, "y"'] \arrow[u, "x"'] & & (\mathrm{sign})^{m-1} \\ & \mathbb{C}x^m \arrow[ul, "y"] \arrow[ur, "x"'] & & & (\mathrm{sign})^m \end{tikzcd}$$

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$$\mathbb{H} = TV \otimes_k kW \Big/ \left(xy - yx = \sum_{w\in W} \langle x,y\rangle_w t_w\right)$$

Fix a basis $x_1, \dots, x_n$ of V. (We were showing....)

Then $\mathbb{H}$ has basis $\{x_{i_1} \cdots x_{i_p} t_w \mid p \in \mathbb{Z}_{\geq 0},\ i_1 \leq \dots \leq i_p,\ w \in W\}$

$\iff$ (a) $\langle wx, wy\rangle_{wvw^{-1}} = \langle x,y\rangle_v$

(b) $\langle x,y\rangle_w (z - wz) + \langle y,z\rangle_w (x - wx) + \langle z,x\rangle_w (y - wy) = 0$

We still need to show if (a), (b) hold then

$$l_x l_y = l_y l_x + \sum_{v\in W} \langle x,y\rangle_v l_v$$

Suppose $i > j \leq i_1$

Then $l_{x_i} l_{x_j} \cdot x_{i_1} \cdots x_{i_p} t_w$

$$= l_{x_i} x_j x_{i_1} \cdots x_{i_p} t_w$$

$$= \left(l_{x_j} l_{x_i} + \sum_{v\in W} \langle x_i, x_j\rangle_v l_v\right) x_{i_1} \cdots x_{i_p} t_w$$

Work also by induction on j. Assume $i > j \geq i_1$.

$$(l_{x_i} l_{x_j} - l_{x_j} l_{x_i}) x_{i_1} \cdots x_{i_p} t_w$$

$$= \left[\left(l_{x_i} (l_{x_{i_1}} l_{x_j} + \sum_{v\in W} \langle x_j, x_{i_1}\rangle_v l_v\right) - l_{x_j}\left(l_{x_{i_1}} l_{x_i} + \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v l_v\right)\right] x_{i_2} \cdots x_{i_p} t_w$$

$$= \Bigg[\left(l_{x_{i_1}} l_{x_i} + \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v l_v\right) l_{x_j} + \sum_{v\in W} \langle x_j, x_{i_1}\rangle_v l_{x_i} l_v$$

$$- \left(l_{x_{i_1}} l_{x_j} + \sum_{v\in W} \langle x_j, x_{i_1}\rangle_v l_v\right) l_{x_i} - \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v l_{x_j} l_v\Bigg] x_{i_2} \cdots x_{i_p} t_w$$

$$= \left[l_{x_{i_1}} \sum_{v\in W} \langle x_i, x_j\rangle_v l_v + \sum_{v\in W} \langle x_i, x_{i_1}\rangle_v l_{vx_j - x_j} l_v - \sum_{v\in W} \langle x_j, x_{i_1}\rangle_v l_{vx_i - x_i} l_v\right] x_{i_2} \cdots x_{i_p} t_w$$

$$= \left[\sum_{v\in W} \langle x_i, x_j\rangle_v l_{x_{i_1}} l_v + \sum_{v\in W} l_{\langle x_i, x_{i_1}\rangle(vx_j - x_j) + \langle x_{i_1}, x_j\rangle(vx_i - x_i)} l_v\right] x_{i_2} \cdots x_{i_p} t_w$$

$$= \left[\sum_{v\in W} \langle x_i, x_j\rangle_v l_{x_{i_1}} l_v + \sum_{v\in W} l_{\langle x_i, x_j\rangle_v (vx_{i_1} - x_{i_1})} l_v\right] x_{i_2} \cdots x_{i_p} t_w$$

$$= \sum_{v\in W} l_{\langle x_i, x_j\rangle_v v x_{i_1}} l_v x_{i_2} \cdots x_{i_p} t_w$$

$$= \sum_{v\in W} \langle x_i, x_j\rangle_v l_v x_{i_1} x_{i_2} \cdots x_{i_p} t_w \quad \blacksquare$$

Now if one had a dependence in $\mathbb{H}$, $\sum a_{i_1 \cdots i_p, w} x_{i_1} \cdots x_{i_p} t_w = 0$

then one would get $0 = \sum a_{i_1 \cdots i_p, w} l_{x_{i_1}} \cdots l_{x_{i_p}} t_w \cdot 1$ in M

$= \sum a_{i_1 \cdots i_p, w} x_{i_1} \cdots x_{i_p} t_w$ in M. Contradiction.

COROLLARY: The PBW Thm holds for $\mathbb{H}$ if

(i) $\langle vx, vy\rangle_{vwv^{-1}} = \langle x,y\rangle_w$

(ii) $\langle\cdot,\cdot\rangle_w = 0$ unless $w=1$ or $\operatorname{codim}(V^w)=2$

and if $\operatorname{codim}(V^w)=2$ then $\operatorname{Rad}\langle\cdot,\cdot\rangle_w \supseteq V^w$

Conversely if $|W| \in k^\times$ then (a), (b) are equivalent to PBW holding.

proof: Assume (i), (ii). Need to prove (b).

If $\langle\cdot,\cdot\rangle_w = 0$ or if $w=1$, (b) is trivially true.

Otherwise, $\operatorname{codim}(V^w)=2$ and $\operatorname{Rad}\langle\cdot,\cdot\rangle_w = V^w$.

If no 2 of x,y,z are lin. indep. mod $\operatorname{Rad}\langle\cdot,\cdot\rangle_w$ then

$\langle x,y\rangle_w, \langle x,z\rangle_w, \langle y,z\rangle_w$ all vanish, so (b) holds.

So assume x,y are lin. indep. mod $\operatorname{Rad}\langle\cdot,\cdot\rangle_w$

and write $z = ax+by$ mod $\operatorname{Rad}\langle\cdot,\cdot\rangle_w$.

$$\langle z,x\rangle_w = b\langle y,x\rangle_w \Rightarrow b = \frac{\langle z,x\rangle_w}{\langle y,x\rangle_w}$$

$$\langle z,y\rangle_w = a\langle x,y\rangle_w \Rightarrow a = \frac{\langle z,y\rangle_w}{\langle x,y\rangle_w}$$

Hence

$$\langle x,y\rangle_w z = \langle z,y\rangle_w x - \langle z,x\rangle_w y \quad \text{mod } \underbrace{\operatorname{Rad}\langle\cdot,\cdot\rangle_w}_{=V^w}$$

$\Big\downarrow$ apply $w-1$

$$\langle x,y\rangle_w(wz-z) = \langle z,y\rangle(wx-x) - \langle z,x\rangle_w(wy-y).$$

Conversely if $|W|\in k^\times$ and we assume (a) & (b),

write $V = V^w \oplus (1-w)V$ (this uses $|W|\in k^\times$)

Claim: $V^w \subseteq \operatorname{Rad}\langle\cdot,\cdot\rangle$ if $w\neq 1$.

proof: If $x,y\in V^w$ then by (b), $\langle x,y\rangle_w(wz-z)=0 \quad \forall z\in V$

$$\Rightarrow \langle x,y\rangle_w = 0.$$

and if $x\in V^w$, $y\in(1-w)V$ say $y=z-wz$

then $\langle x,y\rangle_w = \langle x, z-wz\rangle_w = \langle x,z\rangle_w - \overbrace{\langle x,wz\rangle_w}^{=\langle w^{-1}x,z\rangle_{www^{-1}} = \langle x,z\rangle_w} = 0$ ∎

If $\langle\cdot,\cdot\rangle_w\neq 0$, choose $x,y\in V$ w/ $\langle x,y\rangle_w\neq 0$.

Then by (b), $(wz-z)\in \operatorname{Span}_k(wx-x, wy-y) \quad \forall z\in V$

so $\operatorname{codim} V^w \le 2$ ∎

~~Toy~~ counter-example: $W = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in \mathbb{F}_q \right\} \subset GL(\mathbb{F}_q^2)$

Fixing scalars $c_a \in \mathbb{F}_q \; \forall a \in \mathbb{F}_q$, choose ~~the~~ forms $\langle x,y\rangle = c_a \langle x,y \rangle$

$$= \det \begin{bmatrix} 1 & 1 \\ x & y \\ 1 & 1 \end{bmatrix}$$

Then this satisfies (a) & (b), but fails (i) & (ii).

Real example: $k = \mathbb{C}$, $\mathfrak{h}$ a finite dim'l $\mathbb{C}$-vector space, and a finite group $W \subseteq GL(\mathfrak{h})$ gen'd by reflections. Fix $\kappa \in \mathbb{C}$, $c_s \in \mathbb{C}$ $\forall$ refns $s \in W$ s.t. $c_{wsw^{-1}} = c_s$

Let $V = \mathfrak{h}^* \oplus \mathfrak{h}$ w/ W-action $w(x+y) = wx + wy$

$\forall w \in W$, $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$

Define a collection of forms $\langle \cdot , \cdot \rangle_w$ by:

$$\langle x,y \rangle_1 = \begin{cases} \kappa x(y) & \text{if } x \in \mathfrak{h}^*, y \in \mathfrak{h} \\ 0 & \text{if } x,y \in \mathfrak{h} \text{ or } x,y \in \mathfrak{h}^* \end{cases}$$

and for $w \neq 1$, $\langle x,y \rangle_w = 0$ unless w is a reflection s.

For a reflection $s \in W$, choose $\underset{\mathfrak{h}^*}{\alpha_s} , \underset{\mathfrak{h}}{\alpha_s^\vee}$ so that $sx = x - \langle x, \alpha_s^\vee \rangle \alpha_s$ $(\Leftrightarrow s^{-1}y = y - \langle \alpha_s, y \rangle \alpha_s^\vee)$

and define $$\langle x,y \rangle_s = \begin{cases} c_s \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle & \text{if } x \in \mathfrak{h}^*, y \in \mathfrak{h} \\ 0 & \text{if } x,y \in \mathfrak{h} \text{ or } x,y \in \mathfrak{h}^* \end{cases}$$

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$W \subset GL(\mathfrak{h})$, $\mathfrak{h}$ a fin. dim'l k-vector space

$V = \mathfrak{h}^* \oplus \mathfrak{h}$

A refl'n is an element $s \in W$ s.t. $\operatorname{codim} \mathfrak{h}^s = 1$

$T :=$ refl'ns in W

$\langle \cdot, \cdot \rangle : V \otimes V \to k$ the skew-symmetric form determined by

$$\langle x, y \rangle = \begin{cases} x(y) & \text{if } x \in \mathfrak{h}^*, y \in \mathfrak{h} \\ 0 & \text{if } x, y \in \mathfrak{h}^*, \text{ or } x, y \in \mathfrak{h} \end{cases}$$

For a refl'n $s \in T$, $\exists \alpha_s \in \mathfrak{h}^*$ such that

$$(1-s)x = \langle x, \alpha_s^\vee \rangle \alpha_s \quad \text{for some } \alpha_s^\vee \in \mathfrak{h}$$

In other words, $sx = x - \langle x, \alpha_s^\vee \rangle \alpha_s$ (and $s^{-1}y = y - \langle \alpha_s, y \rangle \alpha_s^\vee$)

Now, fix $\kappa \in k$ and $\{c_s\}_{s \in T} \in k$ such that $c_{wsw^{-1}} = c_s$ $\forall w \in W$

and define skew-symm. forms on $V \otimes V$ by

$$\langle x, y \rangle_1 = -\kappa \langle x, y \rangle$$

$$\langle x, y \rangle_s = \begin{cases} c_s \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle & \text{for } x \in \mathfrak{h}^*, y \in \mathfrak{h} \\ 0 & \text{for } x, y \in \mathfrak{h}^* \text{ or in } \mathfrak{h} \end{cases}$$

$$\langle x, y \rangle_w = 0 \text{ if } w \notin T \cup \{1\}.$$

In order to check that PBW holds for $\mathbb{H}$ $(=: \mathbb{H}_c)$,

we need (i) $\langle wx, wy \rangle_{wvw^{-1}} = \langle x, y \rangle_v$

(ii) $\langle \cdot, \cdot \rangle_w = 0$ unless $w = 1$ or $\operatorname{codim} V^w = 2$

and if $\operatorname{codim} V^w = 2$ then $V^w \subset \operatorname{Rad} \langle \cdot, \cdot \rangle_w$

(ii) is clear. For (i), $\langle \cdot, \cdot \rangle_1$ is W-invariant by defn,

and
$$\begin{aligned} \langle wx, wy \rangle_{wsw^{-1}} &= c_{wsw^{-1}} \langle \alpha_{wsw^{-1}}, wy \rangle \langle wx, \alpha_{wsw^{-1}}^\vee \rangle \\ &= c_s \langle w(\alpha_s), wy \rangle \langle wx, w(\alpha_s^\vee) \rangle \\ &= c_s \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle = \langle x, y \rangle_s \ \checkmark \end{aligned}$$

So PBW holds.

Hence $\mathbb{H} \underset{\substack{\uparrow \\ k\text{-vector space iso.}}}{\cong} S(\mathfrak{h}^*) \otimes_k S(\mathfrak{h}) \otimes_k kW$

$$f(x) f(y) t_w \longleftarrow f(x) \otimes f(y) \otimes t_w$$

LEMMA: For $y \in \mathfrak{h}$ and $S(\mathfrak{h}^*)$

$$[y,f] = yf - fy = \kappa\,\partial_y(f) - \sum_{s\in T} c_s \langle \alpha_s, y\rangle \frac{f - s(f)}{\alpha_s} t_s$$

where ∂_y is the k-linear derivation on $S(\mathfrak{h}^*)$ determined by $\partial_y(x) = \langle x, y\rangle \ \forall x \in \mathfrak{h}^*$

proof: Induct on $\deg(f)$ (for f homogeneous WLOG).

If $\deg(f) = 1$, $f = x \in \mathfrak{h}^*$

$$yx - xy = \underbrace{\kappa\langle x, y\rangle}_{\partial_y(x)} - \sum_{s\in T} c_s \langle \alpha_s, y\rangle \underbrace{\langle x, \alpha_s^\vee\rangle}_{\frac{x - s(x)}{\alpha_s}} t_s \quad ✓$$

Now

$$\begin{aligned}
y(fg) - (fg)y &= (yf - fy)g + f(yg - gy) \\
&= \left(\kappa\partial_y(f) - \sum_{s\in T} c_s\langle\alpha_s, y\rangle \frac{f - s(f)}{\alpha_s} t_s\right)g + f\left(\kappa\partial_y(g) - \sum_{s\in T} c_s\langle\alpha_s, y\rangle\frac{g - s(g)}{\alpha_s} t_s\right) \\
&= \kappa(\partial_y(f)g + f\partial_y(g)) - \sum_{s\in T} c_s \langle\alpha_s, y\rangle\left(\frac{f - s(f)}{\alpha_s} s(g) + f\frac{g - s(g)}{\alpha_s}\right) t_s \\
&= \kappa\,\partial_y(fg) - \sum_{s\in T} c_s\langle\alpha_s, y\rangle \frac{fg - s(fg)}{\alpha_s} t_s \quad \blacksquare
\end{aligned}$$

~~Another version of this~~ is LEMMA: $[g, x] = \kappa\,\partial_x g - \sum_{s\in T} c_s \langle x, \alpha_s^\vee\rangle t_s \frac{g - s^{-1}(g)}{\alpha_s^\vee}$ for $x \in \mathfrak{h}^*$, $g \in S(\mathfrak{h})$ $\blacksquare$

COROLLARY: When $\kappa = 0$, $S(\mathfrak{h})^W \otimes_k S(\mathfrak{h}^*)^W \subset Z(\mathbb{H}_c)$.

Recall for $f \in \mathbb{C}[x_1, \dots, x_n]$ homogeneous, $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \cdot f = (\deg f)\cdot f$

If $\kappa = 1$ and $c_s = 0 \ \forall s \in T$

this says $\sum_{i=1}^n x_i y_i \cdot f = (\deg f)\cdot f$ where $\{x_i\}, \{y_i\}$ are dual bases for $\mathfrak{h}^*, \mathfrak{h}$.

Motivating calculation:

$$\begin{aligned}
\left[\sum_{i=1}^n x_i y_i, x\right] &= \sum_{i=1}^n x_i \left(\kappa\langle x, y_i\rangle - \sum_{s\in T} c_s \langle\alpha_s, y_i\rangle\langle x, \alpha_s^\vee\rangle t_s\right) \\
&= \kappa x - \sum_{s\in T} c_s \alpha_s \langle x, \alpha_s^\vee\rangle t_s \\
&= \kappa x - \sum_{s\in T} c_s (x - s(x)) t_s \\
&= \kappa x - \left[x, \sum_{c\in T} c_s t_s\right]
\end{aligned}$$

This implies $\left[\sum_{i=1}^{n} x_i y_i - \sum_{s \in T} c_s t_s , x \right] = \kappa x$

Define $\mathbb{h} := \sum_{i=1}^{n} x_i y_i + \sum_{s \in T} c_s (1 - t_s)$

Then $[\mathbb{h}, x] = \kappa x$

$\Rightarrow [\mathbb{h}, f] = \kappa \deg(f) \cdot f$ for homogeneous $f \in S(\mathfrak{h}^*)$

because bracketing $[\mathbb{h}, -]$ is a derivation.

Also $[\mathbb{h}, t_w] = 0$

$[\mathbb{h}, y] = -\kappa y$ (similar calculation to above)

<u>COROLLARY</u>: For all $f \in \mathbb{H}$, $[\mathbb{h}, f] = \kappa \cdot \deg(f) \cdot f$

in which $\deg(x) = 1 \quad \forall x \in \mathfrak{h}^*$

$\deg(y) = 1 \quad \forall y \in \mathfrak{h}$

$\deg(t_w) = 0 \quad \forall w \in W$.

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Verma modules for $\mathbb{H}$ for $\lambda \in \mathrm{Irr}\, kW$, k alg. closed

$$M(\lambda) := \mathbb{H} \otimes_{S(\mathfrak{h})\otimes kW} \lambda \qquad \text{with } S(\mathfrak{h})\cdot\lambda = 0 \text{ i.e. } y_i\text{'s kill } \lambda$$

$$S(\mathfrak{h}^*) \otimes_k S(\mathfrak{h}) \otimes_k kW \otimes_{S(\mathfrak{h})\otimes kW} \lambda$$

The $\mathbb{H}$-action on $\lambda \cong S(\mathfrak{h}^*) \otimes_k \lambda$

is given by
$$t_w . fv = w(f)\, wv$$
$$x . fv = xf\, v$$
$$y . fv = \,?$$

Recall $yf - fy = \kappa\, \partial_y f - \sum_{s \in T} \langle \alpha_s, y\rangle \frac{f - sf}{\alpha_s} t_s$

$\Rightarrow$
$$y.fv = \kappa\, \partial_y f v - \sum_{s\in T} c_s \langle \alpha_s, y\rangle \frac{f - sf}{\alpha_s} sv$$

In particular, if $\lambda = \mathrm{triv} = k$

these formulas give an $\mathbb{H}$-action on $S(\mathfrak{h}^*) \otimes_k k = S(\mathfrak{h}^*)$

with
$$y.f = \kappa\, \partial_y f - \sum_{s\in T} c_s \langle \alpha_s, y\rangle \frac{f - s(f)}{\alpha_s}$$

Recall If $\mathbb{h} = \sum_{i=1}^n x_i y_i + \sum c_s (1 - t_s)$ $\qquad$ $x_1, \dots, x_n$, $y_1, \dots, y_n$ dual bases of $\mathfrak{h}^*$, $\mathfrak{h}$

then $[\mathbb{h}, t_w] = 0$
$$[\mathbb{h}, f] = \kappa \deg(f) \cdot f \quad \text{for } f \in S(\mathfrak{h}^*)$$
$$[\mathbb{h}, g] = -\kappa \deg(g) \cdot g \quad g \in S(\mathfrak{h})$$

so
$$\mathbb{h}.fv = ([\mathbb{h}, f] + f\mathbb{h})v$$
$$= \kappa \deg f \cdot fv + f\eta_c v \quad \text{where } \eta_c \in k \text{ is defined}$$
$$= (\kappa \deg f + \eta_c) fv \quad \text{by } \sum c_s(1 - t_s)v = \eta_c v \;\; \forall v \in \lambda$$

COROLLARY: Suppose $\kappa \neq 0$. If k is alg. closed of char 0 and $\lambda \in \mathrm{Irr}\, kW$ then every submodule of $M(\lambda)$ is graded.

proof: Assume $\kappa = 1$. If $N \leq M(\lambda)$ is an $\mathbb{H}$-submodule, it is an $\langle \mathbb{h} \rangle$-submodule, and since $M(\lambda)$ is $\mathbb{h}$-diagonalizable, so is N.

Hence $N = \bigoplus_{a \in k} N_a$ where $N_a = \{ n \in N \mid \mathbb{h}.n = an \}$

Let $M^d(\lambda) := k\text{-span}\{ fv \mid \deg(f) = d,\ v \in \lambda \}$

Then $\mathbb{h}.m = (d + \eta_c)m \quad \forall m \in M^d(\lambda)$

So $N_a = N \cap M^{a-\eta_c}(\lambda)$ ∎

COROLLARY: With the assumptions of the previous corollary, $M(\lambda)$ has a simple top/head, or equivalently a $!$ maximal proper submodule.

proof: Suppose $N \subseteq M(\lambda)$ is an $\mathbb{H}$-submodule.

Then $N \subsetneq M(\lambda) \iff N \cap M^0(\lambda) = 0$

$\Longrightarrow$: $M^0(\lambda) = \lambda$ is kW-irreducible and generates $M(\lambda)$ as $\mathbb{H}$-module.

$\Longleftarrow$: Let $\hat{N} :=$ sum of all proper submodules N

Each $N = \bigoplus_{m>0} N \cap M^m(\lambda) \subset \bigoplus_{m>0} M^m(\lambda)$

so $\hat{N} \subset \bigoplus_{m>0} M^m(\lambda) \subsetneq M(\lambda)$ ∎

S. Griffeth 11/30/06

Let $k=\mathbb{C}$, $W \subset GL(\mathfrak{h})$ a complex refl'n group

Recall $\mathbb{H} = \underbrace{T(\mathfrak{h}\oplus\mathfrak{h}^*)\otimes_{\mathbb{C}} \mathbb{C}W}_{\text{twisted group alg.}} \Big/ \left(yx - xy = \kappa\langle x,y\rangle - \sum_{s\in T} c_s \langle \alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle t_s\right)$

$\forall x \in \mathfrak{h}^*, y\in\mathfrak{h}$

(i.e. $ft_w \cdot gt_v = f\cdot(w(g))\cdot t_{wv}$)

$x_1x_2 = x_2x_1 \quad \forall x_1, x_2 \in \mathfrak{h}^*$

$y_1y_2 = y_2y_1 \quad y_1, y_2 \in \mathfrak{h}$

Fix a W-invariant pos. definite Hermitian form $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ on $\mathfrak{h}$ (conjugate-linear in 2nd variable)

Define $\mathfrak{h} \xrightarrow{*} \mathfrak{h}^*$ (and also $\underset{\text{the inverse map}}{\mathfrak{h}^* \xrightarrow{*} \mathfrak{h}}$)

$y \longmapsto \langle\cdot, y\rangle_{\mathfrak{h}} =: y^*$

$*$ is a conjugate-linear iso. of $\mathbb{R}$-vector spaces

and is W-equivariant

Define a conjugate linear ring anti-homomorphism (if $\underline{\kappa, c_s \in \mathbb{R}}$)

$\mathbb{H} \xrightarrow{*} \mathbb{H}$

by $t_w \overset{*}{\longmapsto} t_w^{-1}$

$x \overset{*}{\longmapsto} x^* \quad \forall x\in\mathfrak{h}^*$

$y \overset{*}{\longmapsto} y^*$

Recall that if $\lambda \in \mathrm{Irr}(\mathbb{C}W)$, then the Verma module

$M(\lambda) := \mathrm{Ind}_{S(\mathfrak{h})\otimes_{\mathbb{C}}\mathbb{C}W}^{\mathbb{H}} \lambda$ where $S(\mathfrak{h})$ acts by the constant term on λ

$\cong S(\mathfrak{h}^*)\otimes_{\mathbb{C}} \lambda$

$\uparrow$ $\mathbb{C}$-vector space iso.

Let $\langle\cdot,\cdot\rangle_\lambda$ be a pos. def. Hermitian W-invariant inner product on λ

Extend it $M(\lambda)$ by $\langle f_1v_1, f_2v_2\rangle_\lambda = \langle ct(f_2^* \cdot f_1v_1), v_2\rangle_\lambda$

for $f_1, f_2 \in S(\mathfrak{h}^*)$, $v_1, v_2 \in \lambda$

where $ct(fv) := f(0)v$ for $f\in S(\mathfrak{h}^*)$, $v\in\lambda$

If f_1, f_2 are homogeneous,

$$\langle f_1v_1, f_2v_2\rangle = \begin{cases} 0 & \text{if } \deg f_1 \neq \deg f_2 \\ \langle f_2^* \cdot f_1v_1, v_2\rangle & \text{if } \deg f_1 = \deg f_2 \end{cases}$$

PROP: $\langle \cdot,\cdot \rangle_\lambda$ is a Hermitian (but possibly degenerate) form on $M(\lambda)$

and $\langle a^* f_1 v_1, f_2 v_2 \rangle_\lambda = \langle f_1 v_1, a f_2 v_2 \rangle_\lambda$

for all $a \in \mathbb{H}$, $f_1, f_2 \in S(\mathfrak{h}^*)$ and $v_1, v_2 \in \lambda$

proof: Hermitian is clear.
Exercise (or 10U) for the rest ∎

PROP: $\mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$ is the unique maximal proper graded submodule of $M(\lambda)$, and is a maximal proper submodule

proof: $\mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$ is graded by definition of $\langle \cdot,\cdot \rangle_\lambda$.

If $f \in \mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$ and $a \in \mathbb{H}$, then $\forall\, g \in M(\lambda)$

$$\langle g, af \rangle_\lambda = \langle a^* g, f \rangle_\lambda = 0 \quad \text{so } af \in \mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$$

i.e. $\mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$ is a submodule.

Let $L := M(\lambda)/\mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$, and we want to show L is simple.

Suppose $0 \neq N \leq L$.

Choose $0 \neq \bar{f} \in N$ and write $\bar{f} = \bar{f}^{top} +$ lower degree terms

Pick g with $\deg(g) = \deg(f^{top})$ and $\langle f, g \rangle_\lambda \neq 0$ (since $f \notin \mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$).

$$\langle \bar{f}, \bar{g} \rangle_\lambda = \langle \bar{f}, \overline{g_1 v_1 + \dots + g_n v_n} \rangle_\lambda = \sum_{i=1}^{n} \langle \overline{g_i^* f}, \bar{v_i} \rangle_\lambda$$

$$\Rightarrow \underset{\overline{g_i^* f^{top}}}{\overset{}{\overline{g_i^* f}}} \neq 0 \text{ for some } i$$

But $\overline{g_i^* f} \in \lambda \cap N$.

Hence since λ is $\mathbb{C}W$-irreducible, $N \supseteq L$, i.e. $N = L$.

It now only remains to show that if $N \subsetneq M(\lambda)$ is a graded submodule, then $N \subset \mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$.

Since N is graded, suffices to show $N^d \subseteq \mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$ $\forall d \in \mathbb{Z}_{\geq 0}$.

$N \cap \lambda = 0$ since N is proper.

Choose some $f \in N^d$: $\langle f, gv \rangle_\lambda = \langle \underbrace{g^* f}_{\in N}, \underset{\lambda}{\underset{\cap}{v}} \rangle_\lambda$ for g homog. $\Rightarrow$ this is 0. So $f \in \mathrm{Rad}\langle \cdot,\cdot \rangle_\lambda$ ∎

Now assume $\kappa = 0$.

Let $M := M(\mathrm{triv}) = M(\mathbb{1})$

$\cong S(\mathfrak{h}^*)$ w/ $\mathbb{H}$-action

$$x.f = xf \qquad \forall\, x \in \mathfrak{h}^*, f \in S(\mathfrak{h}^*)$$

$$t_w f = w(f) \qquad w \in W, y \in \mathfrak{h}$$

$$y.f = \sum_{s \in T} c_s \langle \alpha_s, y \rangle \frac{f - s(f)}{\alpha_s}$$

Hence $I := S(\mathfrak{h}^*)^W_+ \cdot S(\mathfrak{h}^*)$

is an $\mathbb{H}$-submod of M, because of the Leibniz rule

satisfied by $\Delta_s(f) := \dfrac{f - s(f)}{\alpha_s}$:

$$\Delta_s(fg) = \Delta_s(f) \cdot g + s(f) \cdot \Delta_s(g) \quad \text{(easy to check)}$$

$$\Rightarrow \Delta_s(I) \subset I$$

$$\Rightarrow y.I \subset I$$

$$\Rightarrow \mathbb{H}.I \subset I$$

<u>LEMMA</u>: Let η_λ be the scalar by which $\sum_{s \in T} (1 - t_s)$ acts on $\lambda \in \mathrm{Irr}(\mathbb{C}W)$

Then $\eta_\lambda = 0 \iff \lambda = \mathrm{triv}$

<u>proof:</u> $(\Leftarrow)$ is clear.

$(\Rightarrow)$: next time....

12/7/06 >

Change of notation: $V \in \mathrm{Irr}(\mathbb{C}W)$, w/ character χ

Calculate $\chi\left(\sum_{s \in T} (1 - t_s)\right) = |T| \cdot \dim V - \sum_{s \in T} \chi(t_s)$

~~central character~~

$$= |T| \cdot \dim V - \sum_{H \in \mathcal{A}} \sum_{s \in W_H \setminus \{1\}} \chi(t_s)$$

all reflecting hyperplanes for W

Note W_H is cyclic, say $W_H = \langle s_H \rangle$, of order e_H

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of s_H on V.

Then $\displaystyle\sum_{s \in W_H \setminus \{1\}} \chi(t_s) = \sum_{i=1}^{e_H - 1} \lambda_1^i + \ldots + \lambda_n^i$

$$= (e_H - 1) \cdot \dim_{\mathbb{C}} V^{s_H} - \operatorname{codim}_{\mathbb{C}} V^{s_H}$$

So $\displaystyle\chi\left(\sum_{s \in T} (1 - t_s)\right) = |T| \dim V - \sum_{H \in \mathcal{A}} (e_H - 1) \dim V^{s_H} - (\dim V - \dim V^{s_H})$

$$= (|T| + |\mathcal{A}|) \dim V - \sum_{H \in \mathcal{A}} e_H \dim V^{s_H}$$

$$\omega_V\left(\sum_{s\in T}(1-t_s)\right) = \frac{1}{\dim V}\,\chi\left(\sum_{s\in T}(1-t_s)\right)$$

central character $= \frac{1}{\dim V}\,\mathrm{Tr}(-)$ since things in center act by scalars.

$$= |T| + |A| - \sum_{H\in A} e_H \underbrace{\frac{\dim V^{s_H}}{\dim V}}_{\leq 1}$$

≥ 0 w/ equality iff $V = \text{triv.}$, as desired ∎

Note: $0 \leq \omega_V\left(\sum_{s\in T}(1-t_s)\right) \leq |T| + |A|$

Recall $h := \sum_{i=1}^{n} x_i y_i + \sum_{s\in T} c_s(1-t_s)$

$[h, x] = \kappa x$

$[h, f(x)] = \kappa \deg f \cdot f(x)$ $\qquad f(x) \in S(\mathfrak{h}^*)$

$[h, f(y)] = -\kappa \deg f \cdot f(x)$ $\qquad f(y) \in S(\mathfrak{h})$

If $\kappa = 0$, h is central in $\mathbb{H}$,

and if $I = S(\mathfrak{h}^*)^W_+ S(\mathfrak{h}^*)$, then $S(\mathfrak{h}^*)/I$ is an $\mathbb{H}$-module.

Let $R := \mathrm{Rad}(\langle\cdot,\cdot\rangle_{\mathrm{triv}}) \subseteq S(\mathfrak{h}^*) = M(\mathrm{triv})$

(= ! maximal graded submodule of $M(\mathrm{triv})$)

and $M(\mathrm{triv})/R$ is simple.

$\Longrightarrow I \subseteq R$

THM: (Griffeth?) $S(\mathfrak{h}^*)/I$ is $\mathbb{H}$-simple when $\kappa = 0$ and $c_s = +1$ $\forall s \in T$

proof: Need to show $R \subset I$.

It suffices to show $\underbrace{(R^n)^V}_{V\text{-isotypic component inside } R^n} \subseteq I$ $\forall n \geq 0$ and $V \in \mathrm{Irr}\,\mathbb{C}W$

Suppose it fails, and choose n minimal and $V \in \mathrm{Irr}\,\mathbb{C}W$ so that $(R^n)^V \not\subseteq I$

Choose $f \in (R^n)^V$, $f \notin I$.

$$hf.1 = h.f = \sum_{i=1}^{n} \underbrace{x_i y_i.f}_{\substack{\in I \\ \text{by minimality of } n}} + \underbrace{\sum_{s\in T}(1-t_s)f}_{\text{so this must also be in } I}$$

$$\| $$

$$f\underbrace{h.1}_{0 \text{ by direct calculation}} = 0$$

Hence $\sum_{s\in T}(1-t_s)f \in I$
$$\begin{array}{c} \| \\ \omega_V\left(\sum_{s\in T}(1-t_s)\right)\cdot f \end{array}$$

But $f\notin I$, so $\omega_V\left(\sum_{s\in T}(1-t_s)\right)=0 \overset{\text{LEMMA}}{\Longrightarrow} V=\text{triv}.$

But then $f\in (R^n)^V = (R^n)^{\text{triv}} \Rightarrow f\in I$ ↯. ∎

Let's try to understand $S(\mathfrak{h}^*)/I$ as an $\mathbb{H}$-module better, starting with $W = G(r,1,n)$.

$= $ monomial matrices w/ $\sqrt[r]{1}$ entries

Let $\zeta := e^{\frac{2\pi i}{r}}$, $\zeta_i := \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \zeta & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$ ← i^{th} row (and in ith col), s_{ij} = transposition matrix exchanging i,j

The reflections in $G(r,1,n)$

(a) $\zeta_i^{\ell} s_{ij} \zeta_i^{-\ell}$ $\ell=0,1,\dots,r-1$, a single conjugacy class $\alpha_s = x_i - \zeta^{-\ell}x_j$, $\alpha_s^\vee = y_i - \zeta^{\ell}y_j$

(b) $r-1$ conjugacy classes ζ_i^{ℓ}, $\ell=1,2,\dots,r-1$, $\alpha_s = x_i$, $\alpha_s^\vee = (1-\zeta^{-\ell})y_i$

where $y_1,\dots,y_n$ ~~is~~ std. basis for $\mathbb{C}^n = \{\text{column vectors}\}$

$x_1,\dots,x_n$ ——dual basis—— $(\mathbb{C}^n)^* = \{\text{row vectors}\}$

Commutation relation in $\mathbb{H}$:

$$[y,x] = \kappa\langle x,y\rangle - \sum_{s\in T} c_s\langle \alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle t_s$$

If $i<j$,

$$[y_i,x_j] = \kappa\langle x_j,y_i\rangle - c_0\sum_{1\le k<m\le n}\sum_{\ell=0}^{r-1} \overbrace{\langle x_k-\zeta^{-\ell}x_m, y_i\rangle\langle x_j, y_k-\zeta^{\ell}y_m\rangle}^{\text{class (a)}} t_{\zeta_k^{\ell}s_{km}\zeta_k^{-\ell}}$$
$$- \sum_{k=1}^{n}\sum_{\ell=1}^{r-1} \underbrace{c_\ell\langle x_k,y_i\rangle\langle x_j,(1-\zeta^{-\ell})y_k\rangle t_{\zeta_k^{\ell}}}_{\text{class (b)}}$$

where c_0, c_ℓ = rational Cherednik constants

$$= -c_0\sum_{\ell=0}^{r-1} -\zeta^{\ell}t_{\zeta_i^{\ell}s_{ij}\zeta_i^{-\ell}}$$

$$= c_0\sum_{\ell=0}^{r-1}\zeta^{\ell}t_{\zeta_i^{\ell}s_{ij}\zeta_i^{-\ell}}$$

Define $z_i := y_i x_i + c_0 \varphi_i$ where $\varphi_i := \overbrace{\sum_{1 \le j < i} \sum_{l=0}^{r-1} t}^{\text{Jucys-Murphy-type elements in } G(r,1,n)}{}_{\zeta^l s_{ij} \zeta^{-l}}$

<u>THM</u> (Opdam-Dunkl)

$[z_i, z_j] = 0$ and z_i's are simultaneously diagonalizable on $M(\text{triv}) = S(\mathfrak{h}^*)$ for generic parameters.

<u>THM</u> (Dezélée, Etingof-Ginzburg)

The subalgebra gen'd by $\langle z_1, \ldots, z_n, \mathbb{C}G(r,1,n) \rangle$ is isomorphic to the "generalized" graded affine Hecke algebra.